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Determination of the Moduli of Ring Domains by Finite Element Methods

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Abstract

In the present paper we aim to establish a method of finite element approximations by which we can determine the moduli of thin ring domains and thick ones with critical boundary points. Our method matches the abstract definition of Riemann surfaces, and also will offer a new technique of high practical use in numerical calculation. It is characteristic of our method that we adopt ordinary triangular meshes and linear elements on a subregion of an image by a local parameter of every fixed parametric disk, our approximating functions satisfy the boundary conditions exactly even in the case of curvilinear boundary arcs, and express singular property exactly near critical boundary points. Hence the approximations of high precision are obtained, and the fairly good upper and lower bounds to the moduli can be evaluated. It should be noted that we do not adopt any so-called refined or curvilinear mesh near critical boundary points.[1]

Introduction

In the present paper we aim to establish a method of finite element approximations by which we can determine the moduli of thin ring domains and thick ones with critical boundary points (cf. the present authors [10],[11] and [14] for other treatments). Our method matches the abstract definition of Riemann surfaces, and also will offer a new technique of high practical use in numerical calculation.

Let Ω be a ring subdomain of a Riemann surface W whose closure $\overline{\Omega}$ is a compact bordered subregion of W. We assume that two components C_0 and C_1 of the boundary $\partial\Omega$ of Ω consist of piecewise analytic closed curves which satisfy some restricted conditions (see §1.1).

The ring domain Ω can be conformally mapped onto an annulus $A = \{w \mid 1 < |w| < R\}$ for a suitably chosen $R(> 1)$ so that C_0 and C_1 are mapped onto $\Gamma_0 = \{w \mid |w| = 1\}$ and $\Gamma_1 = \{w \mid |w| = R\}$ respectively. Then the *modulus* $M(\Omega)$ of the ring domain Ω is defined by $M(\Omega) = \log R$, which is uniquely determined by Ω. Our aim is to determine $M(\Omega)$.

Let $\mathcal{F}$ be the class of all continuous functions v on $\overline{\Omega}$ with $v = 0$ on C_0 and $v = 1$ on C_1 which satisfy some restricted conditions (see §3.2). Let $\tilde{\mathcal{F}}$ be the class of all locally single-valued continuous functions v on $\overline{\Omega}$ which satisfy the condition $\int_C dv = 1$ for every closed curve C homotopic to C_1 and some restricted conditions. Then the modulus $M(\Omega)$ is characterized by minimal properties

$$(0.1) \qquad \frac{2\pi}{M(\Omega)} = \min_{v\in\mathcal{F}} D(v) \quad \text{and} \quad \frac{M(\Omega)}{2\pi} = \min_{v\in\tilde{\mathcal{F}}} D(v),$$

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[1] The present paper is the complete and unabridged version of [15].

where by $D(v)$ we denote the Dirichlet integral of v over Ω. The functions u and $\tilde{u}$ which minimize the Dirichlet integrals in the classes $\mathcal{F}$ and $\tilde{\mathcal{F}}$ respectively, are called the *harmonic solutions* in $\mathcal{F}$ and $\tilde{\mathcal{F}}$ respectively.

By making use of the relations (0.1), Opfer [17], [18] and Gaier [4] presented methods to obtain upper and lower bounds for the modulus $M(\Omega)$ in the case of some restricted domains Ω (e.g. lattice domains, etc.) by the finite difference approximations. We shall present a method to obtain fairly good upper and lower bounds for $M(\Omega)$ by our finite element approximation even in the case of a thin or thick ring domain Ω with curvilinear boundary arcs, and with critical boundary points.

It is characteristic of our method that we adopt ordinary triangular meshes and linear elements on a subregion of an image by a local parameter of every fixed parametric disk, our approximating functions of u and $\tilde{u}$ satisfy the boundary conditions exactly even in the case of curvilinear boundary arcs, and express singular property exactly near critical boundary points (see §3.2 and §3.3). Hence the approximations of high precision of u and $\tilde{u}$ are obtained, and the fairly good upper and lower bounds to $M(\Omega)$ can be evaluated. It should be noted that we do not adopt any so-called refined or curvilinear mesh near critical boundary points.

§1 is devoted to construction of triangulations K and K' with width h of two kinds. K is a triangulation of $\overline{\Omega}$ and K' is a modification of K.

In §2, we introduce and investigate four classes of element functions on K and K': the *comparable classes* $S = S(K)$ and $\tilde{S} = \tilde{S}(K)$ (with u and $\tilde{u}$ resp.) and the *computable classes* $S' = S'(K')$ and $\tilde{S}' = \tilde{S}'(K')$. $S \subset \mathcal{F}$ and $\tilde{S} \subset \tilde{\mathcal{F}}$, and S' and $\tilde{S}'$ are collections of modifications $v_h' = F(v_h)$ of $v_h \in S$ and $v_h \in \tilde{S}$ respectively, where F defines a one-to-one mapping of S and $\tilde{S}$ onto S' and $\tilde{S}'$ respectively. $D(v_h')$ ($v_h' \in S'$ or $v_h' \in \tilde{S}'$) can be numerically calculated. We shall obtain an estimate

$$D(v_h) \leq D(v_h') + \varepsilon(v_h'), \tag{0.2}$$

where $\varepsilon(v_h')$ is a quantity of $O(h^2)$ which can be explicitly and numerically calculated if v_h' is obtained (see (iii) of Lemma 2.2).

The *finite element approximations* ω_h and u_h' of u in S and S' respectively are defined by the minimalities:

$$D(\omega_h) = \min_{v_h \in S} D(v_h) \quad \text{and} \quad D(u_h') = \min_{v_h' \in S'} D(v_h') \tag{0.3}$$

respectively. u_h' can be obtained by solving a system of linear equations. §3 is devoted to error estimates of ω_h and u_h for u. In Theorems 3.1 and 3.2, we obtain error estimates:

$$D(\omega_h - u) \leq Ch^2 \quad \text{and} \quad D(u_h - u) \leq C'h^2 \quad \text{resp.}, \tag{0.4}$$

where C and C' are constants which depend only on the square integral of 2nd-order partial derivatives of u, the maximum value of partial derivatives of u, the smallest value of interior angles of triangles and transformations of local parameters. Further, in Theorem 3.2, we obtain an estimate for $D(u)$:

$$D(u) \leq D(u_h') + \varepsilon(u_h') \qquad \text{(see (0.2))}. \tag{0.5}$$

The *finite element approximations* $\tilde{\omega}_h$ and $\tilde{u}_h'$ of $\tilde{u}$ in $\tilde{S}$ and $\tilde{S}'$ respectively are defined by the method analogous to (0.3), and the estimates analogous to (0.4) and (0.5) hold.

Finally, in §4 we apply our results to numerical calculation of the moduli of thin ring domains and thick ones, and we shall show that calculation results for some concrete ring domains are fairly good. With respect to the problems of this type, there have been some investigations by means of the Fourier series method (cf. Gaier and Papamichael [9]), the integral equation method (cf. Symm [23]), the modified Schwarz-

Christoffel transformation method (cf. Gaier [6], Howell and Trefethen [12]), the domain decomposition method (cf. Gaier and Hayman [7], [8], Papamichael and Stylianopoulos [20], [21], [22]), the finite difference method (cf. Gaier [4], [5], Mizumoto [13], Opfer [17], [18]) and the finite element method (cf. Hara and Mizumoto [10], [11], Mizumoto and Hara [14], Weisel [25], [26], [27]).

Numerical treatments of singularities of functions at critical boundary points have been studied by many authors. Especially, with respect to treatments related to our method we can refer Babuška, Szabo and Katz [1], Barnhill and Whiteman [2], Opfer and Puri [19], and Thatcher [24].

§1. Triangulation

1.1 Collection Φ of local parameters. Let Ω be a ring subdomain of a Riemann surface W whose closure $\overline{\Omega}$ is a compact bordered subregion of W and whose boundary $\partial\Omega$ consists of two piecewise analytic closed curves C_0 and C_1. Let $\{p_n\}_{n=1}^{\nu}$ be the collection of the *critical points* defined as the vertices on $\partial\Omega$ at which the curves $\partial\Omega$ are not analytic. Then we assume that there exist parametric disks V_n $(n = 1, \cdots, \nu)$ with the centers p_n and the local parameters $z = \psi_n(p)$ by which $V_n \cap \overline{\Omega}$ are mapped onto sectors $\{|z| < r_n\} \cap \{0 \leq \arg z \leq \beta_n\}$ $(0 < \beta_n \leq 2\pi, \beta_n \neq \pi)$.

By $\Phi = \{z = \varphi_j(p),\ U_j;\ j = 1, \cdots, m\}$ we denote a finite collection of local parameters $z = \varphi_j(p)$ $(j = 1, \cdots, m)$ and parametric disks U_j $(j = 1, \cdots, m)$ on W which satisfies the following conditions (i) $\sim$ (vi):

(i) By the mapping $z = \varphi_j(p)$ $(j = 1, \cdots, m)$, U_j is mapped onto a disk $|z| < \rho_j$.

(ii) $\overline{\Omega}$ is covered by $\{U_j\}_{j=1}^{m}$.

(iii) If $U_j \cap U_k \neq \emptyset$, then there exists a point $p_\iota \in U_j \cap U_k$ such that for the mapping $\zeta = \varphi(z) \equiv \varphi_k \circ \varphi_j^{-1}(z)$, the normalization condition $|d\varphi(z_\iota)/dz| = 1$ $(z_\iota = \varphi_j(p_\iota))$ is satisfied.

(iv) Each U_j $(j = 1, \cdots, m)$ contains at most one p_n $(n = 1, \cdots, \nu)$ and if $p_n \in U_j$ then $\varphi_j(p_n) = 0$.

(v) If $U_j \cap \partial\Omega \neq \emptyset$ and U_j does not contain any p_n $(n = 1, \cdots, \nu)$, then $\varphi_j(U_j \cap \Omega)$ is a half disk $\{|z| < \rho_j\} \cap \{\operatorname{Im} z > 0\}$. If U_j contains some p_n $(n = 1, \cdots, \nu)$, then $\varphi_j(U_j \cap \Omega)$ is a sector $\{|z| < \rho_j\} \cap \{0 < \arg z < \alpha_j\}$ $(0 < \alpha_j \leq 2\pi,\ \alpha_j \neq \pi)$.

(vi) In the latter case of (v), by the mapping $\zeta = (\varphi_j(p))^{\pi/\alpha_j}$, $U_j \cap \Omega$ is mapped onto a half disk $\{|\zeta| < \rho_j^{\pi/\alpha_j}\} \cap \{\operatorname{Im} \zeta > 0\}$. In this case we define anew $z = \varphi_j(p)$ and ρ_j by $\zeta = (\varphi_j(p))^{\pi/\alpha_j}$ and ρ_j^{π/α_j} respectively. Then, the local parameter $z = \varphi_j(p)$ is no longer conformal at the center of U_j.

1.2. Triangulation K associated to Φ. For the collection Φ of local parameters and parametric disks defined in §1.1, and for a sufficiently small positive number h, we construct a triangulation $K = K^h$ of $\overline{\Omega}$ which satisfies the following conditions (i) $\sim$ (v). This is called a *triangulation of $\overline{\Omega}$ with width h associated to Φ.*

(i) The points $p_1, \cdots, p_\nu$ are carriers of some 0-simplices of K.

(ii) K is the sum of subtriangulations $K_1, \cdots, K_m$ of K such that each 2-simplex of K belongs to one and only one K_j $(j = 1, \cdots, m)$, the carrier $|s|$ of each 2-simplex s of K_j is contained in U_j, and $|K| = \overline{\Omega}$.

(iii) If a 1-simplex $e \in K_j$ does not belong to another $K_k (k \neq j)$, or a 1-simplex e belongs to $K_j \cap K_k$ $(j \neq k)$ and the mapping $\varphi_k \circ \varphi_j^{-1}$ is an affine transformation, then $\varphi_j(e)$ is a segment and e is said to be *linear.*

If $\varphi_j(s)$ for $s \in K_j$ $(j = 1, \cdots, m)$ is an ordinary triangle, then s is called a *natural simplex.* Then, by (iii), each 2-simplex $s \in K_j$ which has not a common edge with any 2-simplex of another K_k $(k \neq j)$, is a natural simplex.

A 2-simplex of K_k which has a common edge with a 2-simplex $s \in K_j$ $(j \neq k)$ is said to be an *adjoint* (simplex) of s and is denoted by s'.

(iv) For each pair of a 2-simplex $s \in K_j$ and its adjoint $s' \in K_k$ with a common edge e, either one of the following three cases (a), (b) and (c) occurs.

(a) All edges of s and s' are linear, and thus both s and s' are natural simplices.

(b) $\varphi_j(s)$ is a curvilinear triangle such that $\varphi_j(e)$ is a strictly concave arc w.r.t. $\varphi_j(s)$, s' is a natural simplex, and all edges of s and s' except for e are linear (cf. Fig. 1). Then s is called a *minor simplex.* The case where s' is a minor simplex and s is its adjoint may also occur.

(c) $\varphi_j(s)$ is a curvilinear triangle such that $\varphi_j(e)$ is a strictly convex arc w.r.t. $\varphi_j(s)$, s' is a natural simplex, and all edges of s and s' except for e are linear (cf. Fig. 2). Then s is called a *major simplex.* The case where s' is a major simplex and s is its adjoint may also occur.

If s is a minor or major simplex of K_j, then it is assumed that $|s'| \subset U_j$ for its adjoint s'.

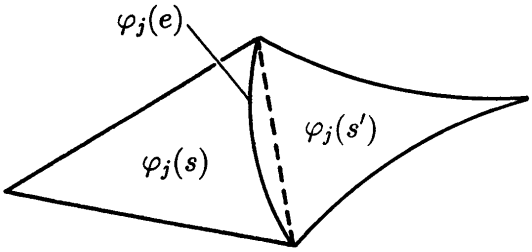

Fig. 1 Minor simplex s and its adjoint s'

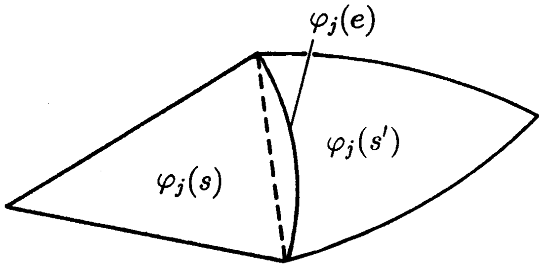

Fig. 2 Major simplex s and its adjoint s'

(v) For each 2-simplex $s \in K_j$ $(j = 1, \cdots, m)$, $d(\varphi_j(s)) \le h$, where throughout the present paper we denote the diameter of a region G by $d(G)$.

Next, we assume that for the fixed Φ the class of the triangulations $K = K^h$ satisfies the following conditions (i$'$) and (ii$'$):

(i$'$) For each $j = 1, \cdots, m$, the union of carriers of all minor and major simplices of K_j, and all their adjoints is contained in a closed subset R_j of $U_j \cap \overline{\Omega}$ which is independent of the individual triangulation K.

(ii$'$) The number N of minor and major simplices of K satisfies the inequality: $N \le M/h$, where M is a constant which is independent of the individual triangulation K.

1.3. Normal subdivision of triangulation K. For a triangulation $K = K^h$ of $\overline{\Omega}$ with width h associated to Φ we can construct a subdivision $K^1 = K^{1,\,h/2}$, called the *normal subdivision* of $K = K^h$ by the following procedure:

(i) K^1 is the sum of the subtriangulations $K_1^1, \cdots, K_m^1$ which are the subdivisions of $K_1, \cdots, K_m$ respectively which are defined in the following (ii) and (iii).

(ii) If $s \in K_j$ is a natural simplex, then s is subdivided to four 2-simplices s_1, s_2, s_3 and s_4 of K_j^1 so that $\varphi_j(s_1), \varphi_j(s_2), \varphi_j(s_3)$ and $\varphi_j(s_4)$ are mutually congruent ordinary triangles as in Fig. 3.

(iii) Let $s \in K_j$ and $s' \in K_k$ be a minor (or major) simplex and its adjoint respectively with a common

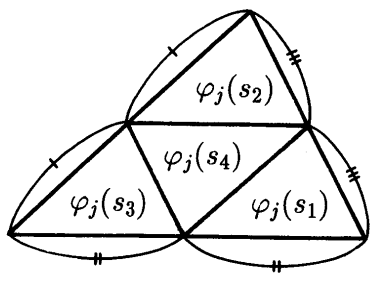

Fig. 3 Normal subdivision of natural simplex

edge e. We subdivide the edge e to two edges e_1 and e_2 so that $\varphi_k(e_1)$ and $\varphi_k(e_2)$ have the same length. Then we subdivide the simplex s to two minor (or major resp.) simplices s_1 and s_2 of K_j^1 and, two natural simplices s_3 and s_4 of K_j^1 as in Fig. 4. Here we note that such a subdivision is always possible if h is sufficiently small.

We can see that the normal subdivision $K^1 = \sum_{j=1}^{m} K_j^1$ is a triangulation of $\overline{\Omega}$ with width $h/2 + O(h^2)$ associated to Φ (cf. §1 of [14]).

1.4. Naturalized triangulation. For each minor (or major) simplex $s \in K_j$ we define the *naturalized simplex* $\natural s$ of s as the 2-simplex such that $|s| \subset |\natural s|$ ($|\natural s| \subset |s|$ resp.) and $\varphi_j(\natural s)$ is the ordinary triangle which has two common sides with $\varphi_j(s)$. Further we define a 2-simplex $\flat\ell = \flat\ell(s)$ ($\sharp\ell = \sharp\ell(s)$ resp.) with two edges whose carrier is the closed region $\overline{|\natural s| - |s|}$ ($\overline{|s| - |\natural s|}$ resp.). $\flat\ell(s)$ ($\sharp\ell(s)$ resp.) is called the *deficient* (*excessive* resp.) *lune* of s.

Each triple of a minor (or major) simplex $s \in K_j$, its adjoint $s' \in K_k$ and its deficient lune $\flat\ell$ (excessive lune $\sharp\ell$ resp.) is denoted by $(s, s', \flat\ell)$ ($(s, s', \sharp\ell)$ resp.), and is called a *triple for a minor* (*major* resp.) *simplex* s or *for a deficient* (*excessive* resp.) *lune* $\flat\ell$ ($\sharp\ell$ resp.) (cf. Fig. 5), where it is assumed that $|\flat\ell| \subset |s'|$ for each $(s, s', \flat\ell)$ whose condition is always satisfied if h is sufficiently small. For simplicity of notation, we also denote $\flat\ell = \flat\ell(s)$ or $\sharp\ell = \sharp\ell(s)$ by $\ell = \ell(s)$. If a minor or major simplex s is in K_j, then we say that $\ell = \ell(s)$ is a *lune of* K_j and write $\ell \in K_j$.

Now we shall define the *naturalized triangulation* K' *associated to* K. First, K_j' ($j = 1, \cdots, m$) are defined as triangulations such that the collection of all 2-simplices of K_j' consists of all natural 2-simplices of K_j, and of all naturalized simplices of minor or major ones of K_j. Then the triangulation K' is defined as the sum of K_j' ($j = 1, \cdots, m$). We should note that K' is no longer a triangulaton of $\overline{\Omega}$, and also is not an ordinary triangulation.

1.5. Parametrization of lunar domains. Let (s, s', ℓ) be a triple for an arbitrary deficient or excessive lune ℓ, and let e_1 and e_2 be two edges of ℓ so oriented that $e_1 \subset \partial(\natural s)$ and $e_2 \subset \partial s$. Further, let

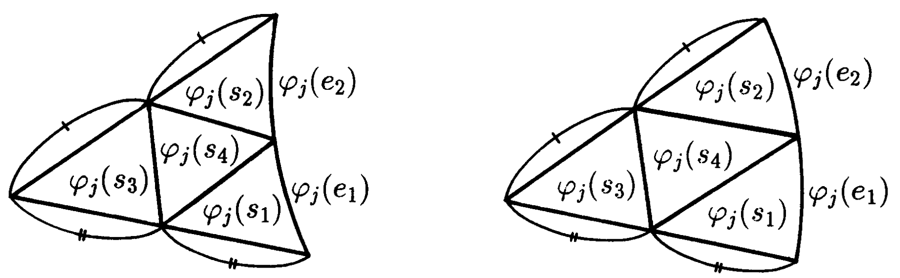

Fig. 4 Normal subdivision of minor simplex and major one. e_1 and e_2 are so determined that $\varphi_k(e_1)$ and $\varphi_k(e_2)$ have the same length.

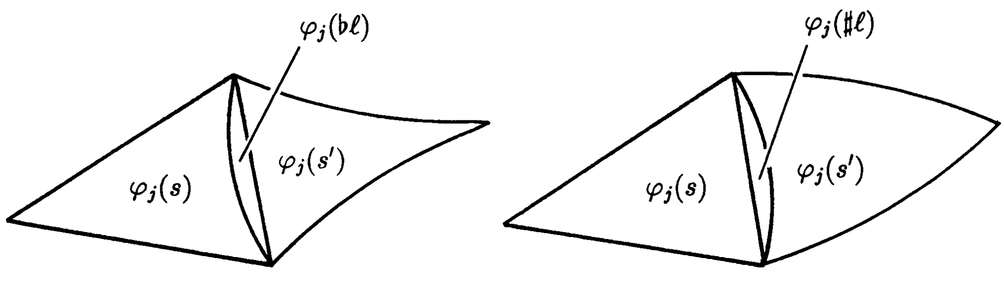

Fig. 5 Triple $(s, s', \flat\ell)$ for a minor simplex and triple $(s, s', \sharp\ell)$ for a major simplex

$z' = (1-t)z_1 + tz_2$ and $\zeta'' = (1-t)\zeta_1 + t\zeta_2$ $(0 \le t \le 1)$ be parameter representations of the oriented segments $\varphi_j(e_1)$ and $\varphi_k(e_2)$ respectively. The last representation induces a parameter representation of the curve $\varphi_j(e_2)$: $z'' = \psi((1-t)\zeta_1 + t\zeta_2)$ $(0 \le t \le 1)$, where $z = \psi(\zeta) \equiv \varphi_j \circ \varphi_k^{-1}(\zeta)$. Then we obtain a parameter representation of the lunar domain $\varphi_j(\ell)$:

$$
\begin{aligned}
(1.1) \qquad z &= z(t,\tau) \equiv (1-\tau)z' + \tau z'' \\
&= (1-\tau)((1-t)z_1 + tz_2) + \tau\,\psi((1-t)\zeta_1 + t\zeta_2) \quad (0 \le t \le 1,\ 0 \le \tau \le 1).
\end{aligned}
$$

1.6. Area of lune.

Lemma 1.1. *Let (s, s', ℓ) be a triple for an arbitrary deficient or excessive lune ℓ. Then, the following estimate holds:*

$$
(1.2) \qquad A(\varphi_j(\ell)) \le \frac{h_1^3}{8}\left(\left|\frac{\psi''(\zeta_1)}{\psi'(\zeta_1)^2}\right| + \kappa h_1\right),
$$

where throughout the present paper we denote the area of a region G by $A(G)$, $z = \psi(\zeta) \equiv \varphi_j \circ \varphi_k^{-1}(\zeta)$, $h_1 = d(\varphi_j(\ell))$, ζ_1 is one of the vertices of the lunar domain $\varphi_k(\ell)$ and $\kappa > 0$ is a constant depending only on ψ.

See Lemma 1.1 of [14] for the proof.

§2. Classes of functions

2.1 Class $\mathcal{G}$. By $\mathcal{G}$ we denote the class of all locally single-valued continuous functions v on $\overline{\Omega} = \Omega \cup \partial\Omega$, for which the partial derivatives $\partial v/\partial x$ and $\partial v/\partial y$ with respect to the local parameter $z = x + iy$ exist and are continuous on Ω at most except for a finite number of rectifiable curves on Ω, and for which the Dirichlet integral $D(v) = D_\Omega(v)$ is finite, where the Dirichlet integral $D_G(v)$ is defined by $D_G(v) \equiv \iint_\Omega \left((\partial v/\partial x)^2 + (\partial v/\partial y)^2\right) dxdy$ for each subregion G of $\overline{\Omega}$.

2.2. Subclass Σ of $\mathcal{G}$. We define a subclass $\Sigma = \Sigma(K)$ of $\mathcal{G}$, called the *comparable class* (with u), as the class of functions v_h which satisfy the following conditions (i) $\sim$ (iv):

(i) $v_h \in \mathcal{G}$.

(ii) If $s \in K_j$ $(j = 1, \cdots, m)$ is a natural simplex which is not an adjoint of any minor or major simplex, then v_h is a linear expression $ax + by + c$ $(a, b, c$: constants) of local variables x and y $(z = \varphi_j(p) = x + iy)$ on $\varphi_j(s)$.

(iii) Let $(s, s', \flat\ell)$ be a triple for a minor simplex s. Then v_h is a linear expression of local variables on each of $\varphi_j(s)$ and $\varphi_k(s') - \varphi_k(\flat\ell)$ respectively, and harmonic in $|\flat\ell|$.

(iv) Let $(s, s', \sharp\ell)$ be a triple for a major simplex s. Then v_h is a linear expression of local variables on each of $\varphi_j(\natural s)$ and $\varphi_k(s')$ respectively, and harmonic in $|\sharp\ell|$.

2.3. Class Σ' of functions. Let K' be the naturalized triangulation associated to K. For each function $v_h \in \Sigma$, we define the *function v'_h on K' associated to v_h* as the function v'_h which satisfies the following conditions (i) and (ii):

(i) For each 2-simplex $s \in K'_j$ $(j = 1, \cdots, m)$, v'_h is a linear expression of local variables on $\varphi_j(s)$.

(ii) $v'_h = v_h$ on the carrier of K minus all lunes.

We should note that the function v'_h is defined just twice on each deficient lune $\flat\ell$, while it is never defined on any excessive lune $\sharp\ell$. Hereafter in the former case, for each triple $(s, s', \flat\ell)$ we shall denote the branches of v'_h on $\natural s \in K'_j$ and $s' \in K'_k$ by v'_{hs} and $v'_{hs'}$ respectively.

The class of all functions v'_h associated to $v_h \in \Sigma$ is denoted by $\Sigma' = \Sigma'(K')$ and called the *computable class*. Let L' be a subcomplex of K', and let v'_h and ψ'_h be functions in Σ'. The *mixed Dirichlet integral*

$D_{L'}(v'_h, \psi'_h)$ of v'_h and ψ'_h over L' and the *Dirichlet integral* $D_{L'}(v'_h)$ of v'_h over L' are defined by [2]

$$D_{L'}(v'_h, \psi'_h) \equiv \sum_{s \in L'} \iint_{|s|} \left(\frac{\partial v'_h}{\partial x}\frac{\partial \psi'_h}{\partial x} + \frac{\partial v'_h}{\partial y}\frac{\partial \psi'_h}{\partial y} \right) dxdy \qquad \text{and}$$

$$D_{L'}(v'_h) \equiv D_{L'}(v'_h, v'_h), \qquad \text{resp.}$$

Further, for simplicity we set $D(v'_h, \psi'_h) = D_{K'}(v'_h, \psi'_h)$ and $D(v'_h) = D_{K'}(v'_h)$, where $D(v'_h)$ can be numerically calculated if v'_h is obtained. We see that $v'_h = F(v_h)$ defines a one-to-one mapping of Σ onto Σ'.

2.4. Finite element interpolations. Let v be a function of $\mathcal{G}$. We define the *finite element interpolation* $\widehat{v}$ *of* v in the class Σ as the function uniquely determined by the following conditions (i) and (ii):

(i) $\widehat{v} \in \Sigma$;

(ii) $\widehat{v}(p) = v(p)$ at the carrier p of each 0-simplex of K.

2.5. Harmonic functions on a lune.

Lemma 2.1. *Let $\ell = \ell(s)$ be a deficient or excessive lune of K_j, let e_1 and e_2 be the edges of ℓ so oriented that $e_1 \subset \partial(\natural s)$ and $e_2 \subset \partial s$, and q_1 and q_2 be the vertices of ℓ such that $\partial e_1 = \partial e_2 = q_2 - q_1$. Let λ and μ be the length of the segments $\varphi_j(e_1)$ and $\varphi_k(e_2)$ respectively, and let ϑ (and δ) be the angle between the oriented segment $\varphi_j(e_1)$ ($\varphi_k(e_2)$ resp.) and the x-axis (the ξ-axis resp.), where $z = \varphi_j(p) = x + iy$ and $\zeta = \varphi_k(p) = \xi + i\eta$.*

Let H be the harmonic function in ℓ which is continuous on ℓ, and satisfies the boundary conditions: $H = ax + by + c$ on $\varphi_j(e_1)$ and $H = \alpha\xi + \beta\eta + \gamma$ on $\varphi_k(e_2)$, where a, b, c, α, β and γ are constants. When we set

$$\varepsilon_\ell(H) \equiv A(\varphi_j(\ell)) \cdot \left(\frac{H(q_2) - H(q_1)}{\mu} \right)^2 \cdot \max_{\varphi_j(\ell)} \frac{|\varphi'(z)|^4}{(\mathrm{Re}(e^{i(\vartheta - \delta)}\varphi'(z)))^2}, \tag{2.1}$$

the following inequality holds

$$D_\ell(H) \leq \varepsilon_\ell(H), \tag{2.2}$$

where $D_\ell(H) \equiv \iint_{|\ell|}((\partial H/\partial x)^2 + (\partial H/\partial y)^2)dxdy$, $\zeta = \varphi(z) \equiv \varphi_k \circ \varphi_j^{-1}(z)$ and hereafter the notation (2.1) is preserved. Further the inequalities

$$\varepsilon_\ell(H) \leq A(\varphi_j(\ell)) \cdot \left(\frac{H(q_2) - H(q_1)}{\lambda} \right)^2 \cdot (1 + \kappa h) \leq A(\varphi_j(\ell))(a^2 + b^2)(1 + \kappa h) \tag{2.3}$$

hold, where by $\kappa > 0$ we denote a constant depending only on the local mapping $\zeta = \varphi(z)$ and hereafter the notation is preserved.

Proof. We make use of the parameter representation (1.1) of the lunar domain $\varphi_j(\ell)$ and we preserve the notations in §1.5. Without loss of generality we may assume that the oriented segments $\varphi_j(e_1)$ and $\varphi_k(e_2)$ lie on the real axes $\mathrm{Im}\, z = 0$ and $\mathrm{Im}\, \zeta = 0$ respectively, and $z_1 = 0$, $z_2 = \lambda$ ($\lambda > 0$), $\zeta_1 = 0$ and $\zeta_2 = \mu$ ($\mu > 0$). Then the harmonic function H satisfies the boundary conditions: $H = ax + c$ on $\varphi_j(e_1)$ and $H = \alpha \mathrm{Re}\, \varphi(z) + \gamma$ on $\varphi_j(e_2)$, and $H(q_1) = c = \gamma$ and $H(q_2) - H(q_1) = a\lambda = \alpha\mu$.

Let $\Theta = \Theta(z)$ be the functon on $\varphi_j(\ell)$ obtained by setting $\Theta(z) = a\lambda t + c$ at a point $z = z(t, \tau) \in \varphi_j(\ell)$. Since the function $\Theta \circ \varphi_j$ satisfies the common boundary condition as H and H is harmonic in ℓ, the inequality $D_\ell(H) \leq D_\ell(\Theta \circ \varphi_j)$ holds. Further, when we note that $\mathrm{Re}\, \psi'(\zeta) > 0$ ($z = \psi(\zeta) \equiv \varphi^{-1}(\zeta)$) on

[2] We shall use the common notations $D(\ ,\ )$ and $D(\ )$ for both mixed and ordinary Dirichlet integrals of functions of the classes $\mathcal{G}$ and Σ'.

$\varphi_k(\ell)$ for a sufficiently small h, we can verify that

$$|\text{grad}\,\Theta| \;=\; |a|\,\lambda\,|\text{grad}\,t| \;=\; \frac{|a|\,\lambda}{|(1-\tau)\lambda+\tau\mu\,\psi'(\mu t)|} \;\le\; \max\left\{|a|,\ \frac{|\alpha|}{\text{Re}\,\psi'(\mu t)}\right\}$$

$$\le\; \max\left\{|a|,\ |\alpha| \max_{z\in\varphi_j(e_2)} \frac{|\varphi'(z)|^2}{|\text{Re}\,\varphi'(z)|}\right\} \;\le\; |\alpha| \max_{\varphi_j(\ell)} \frac{|\varphi'(z)|^2}{|\text{Re}\,\varphi'(z)|}.$$

Therefore the inequality (2.2) is obtained.

By making use of the power series expansion of φ' around a vertex $z=0$ of the lunar domain $\varphi_j(\ell)$ and the relation $\mu=\varphi(\lambda)$, we can verify that $\max_{\varphi_j(\ell)}|\varphi'(z)|\le|\varphi'(0)|(1+\kappa h)$, $\max_{\varphi_j(\ell)}|\varphi'(z)/\text{Re}\,\varphi'(z)|\le |\varphi'(0)/\text{Re}\,\varphi'(0)|(1+\kappa h)$, $\mu\ge(|\varphi'(0)|-\kappa h)\int_{\varphi_j(e_1)}|dz|=\lambda(|\varphi'(0)|-\kappa h)$ and $|\varphi'(0)/\text{Re}\,\varphi'(0)|\le 1+\kappa h$ hold. Therefore we have the inequality

$$|\alpha| \max_{\varphi_j(\ell)} \frac{|\varphi'(z)|^2}{|\text{Re}\,\varphi'(z)|} \;\le\; |a|\,(1+\kappa h). \tag{2.4}$$

(2.1) and (2.4) imply the inequality (2.3).

Lemma 2.2. *Let v_h be a function of the class Σ.*

(i) *For each $\sharp\ell\in K_j$ and its triple $(s,s',\sharp\ell)$ the inequalities*

$$D_{\sharp\ell}(v_h) \;\le\; \varepsilon_{\sharp\ell}(v_h') \;\le\; A(\varphi_j(\sharp\ell))\,|(\nabla v_h')_{\natural s}|^2\,(1+\kappa h) \tag{2.5}$$

hold, where hereafter by $(\nabla v_h')_{\natural s}$ we denote the constant gradient of v_h' on $\varphi_j(\natural s)$.

(ii) *If for each $\flat\ell\in K_j$ and its triple $(s,s',\flat\ell)$ we set $H=v_{hs}'+v_{hs'}'-v_h$ on $\flat\ell$ (see §2.3 for the notations), then the inequalities*

$$D_{\flat\ell}(H)\le\varepsilon_{\flat\ell}(v_h') \;\le\; A(\varphi_j(\flat\ell))\,|\nabla v_{hs}'|^2\,(1+\kappa h) \tag{2.6}$$

hold, where hereafter by $\nabla v_{hs}'$ we denote the constant gradient of v_{hs}' on $\varphi_j(\natural s)$.

(iii) *The inequality*

$$D(v_h) \;\le\; D(v_h')+\varepsilon(v_h') \tag{2.7}$$

holds, where

$$\begin{aligned}\varepsilon(v_h') \;\equiv\; &\sum_{\sharp\ell\in K}\varepsilon_{\sharp\ell}(v_h')+\sum_{\flat\ell\in K}\Big(\varepsilon_{\flat\ell}(v_h')\\ &+2\left(D_{\flat\ell}(v_{hs}')^{1/2}D_{\flat\ell}(v_{hs'}')^{1/2}+D_{\flat\ell}(v_{hs}')^{1/2}\varepsilon_{\flat\ell}(v_h')^{1/2}+D_{\flat\ell}(v_{hs'}')^{1/2}\varepsilon_{\flat\ell}(v_h')^{1/2}\right)\Big)\end{aligned} \tag{2.8}$$

and $\varepsilon(v_h')=O(h^2)$. Further by the definition (2.1) of $\varepsilon_{\sharp\ell}(v_h')$ and $\varepsilon_{\flat\ell}(v_h')$, and the equalities $D_{\flat\ell}(v_{hs}')=A(\varphi_j(\flat\ell))|\nabla v_{hs}'|^2$ and $D_{\flat\ell}(v_{hs'}')=A(\varphi_k(\flat\ell))|\nabla v_{hs'}'|^2$, we see that $\varepsilon(v_h')$ is a quantity which can be numerically calculated if v_h' is obtained.

Proof. (i) Since the fuction v_h on $\sharp\ell$ has the common property as the function H of Lemma 2.1, the inequalities (2.5) hold.

(ii) Since the function H has the common property as H of Lemma 2.1, and $H(q_1)=v_h'(q_1)$ and $H(q_2)=v_h'(q_2)$, the inequalities (2.6) hold.

(iii) By the definition of v_h and v_h' the equality

$$D(v_h)-D(v_h') \;=\; \sum_{\sharp\ell\in K}D_{\sharp\ell}(v_h)+\sum_{\flat\ell\in K}\Big(D_{\flat\ell}(v_h)-D_{\flat\ell}(v_{hs}')-D_{\flat\ell}(v_{hs'}')\Big) \tag{2.9}$$

holds and further, for each triple $(s,s',\flat\ell)$ the inequality

$$D_{\flat\ell}(v_h)^{1/2} \;\le\; D_{\flat\ell}(v_{hs}')^{1/2}+D_{\flat\ell}(v_{hs'}')^{1/2}+D_{\flat\ell}(H)^{1/2} \tag{2.10}$$

holds. (2.9), (2.10), (2.5) and (2.6) imply the inequality (2.7). Further by Lemma 1.1 and (ii′) of §1.2, $\varepsilon(v'_h) = O(h^2)$ is obtained.

§3. Finite element approximations

3.1 Formulation of problems. The ring domain Ω defined in §1.1 can be conformally mapped onto an annulus $A = \{w \mid 1 < |w| < R\}$ for a suitably chosen number R so that C_0 and C_1 are mapped onto $\Gamma_0 = \{w \mid |w| = 1\}$ and $\Gamma_1 = \{w \mid |w| = R\}$ resp. Then the *modulus* $M(\Omega)$ of the ring domain Ω is defined by $M(\Omega) = \log R$, which is uniquely determined by Ω. Our aim is to determine the modulus $M(\Omega)$ of a ring domain Ω by our finite element methods.

3.2. Classes $\mathcal{F}$ and $\tilde{\mathcal{F}}$. By $\mathcal{F}$ we denote the subclass of $\mathcal{G}$ defined in §2.1 which consists of all single-valued functions v on $\overline{\Omega}$ satisfying the boundary conditions $v = 0$ on C_0 and $v = 1$ on C_1. Further, by $\tilde{\mathcal{F}}$ we denote the subclass of $\mathcal{G}$ which consists of all locally single-valued functons v on $\overline{\Omega}$ satisfying the condition $\int_C dv = 1$ for every closed curve C homotopic to C_1.

By u we denote the uniquely determined function of $\mathcal{F}$ which is harmonic on Ω. Also, by $\tilde{u}$ we denote the function of $\tilde{\mathcal{F}}$ which is harmonic on Ω and satisfies the condition $*d\tilde{u} = 0$ along $\partial\Omega$, where by $*d\tilde{u}$ we denote the conjugate differential of $d\tilde{u}$. The function $\tilde{u}$ is uniquely determined except for an additive constant.

By (vi) of §1.1, and the boundary conditions of the functions u and $\tilde{u}$, they can be harmonically continued over $\varphi_j(U_j \cap \partial\Omega)$ on each $\varphi_j(U_j)$ even in the case where some critical point p_n belongs to U_j. Hence, by the transformation of local parameter of (vi) of §1.1 all singularities of u and $\tilde{u}$ at the critical points of $\partial\Omega$ vanish.

3.3. Minimalities. The modulus $M(\Omega)$ is characterized by the following minimal properties, which can be proved by standard arguments using Green's formula.

Lemma 3.1. *The equalities*

$$\frac{2\pi}{M(\Omega)} = D(u) = \min_{v\in\mathcal{F}} D(v) \quad \textit{and} \quad \frac{M(\Omega)}{2\pi} = D(\tilde{u}) = \min_{v\in\tilde{\mathcal{F}}} D(v) \tag{3.1}$$

hold. The both minimums of the right hand sides of (3.1) *are attained if and only if* $v = u$ *and* $v = \tilde{u}+\text{const.}$ *respectively.*

We call u and $\tilde{u}$ the *harmonic solutions* in $\mathcal{F}$ and $\tilde{\mathcal{F}}$ respectively. Let $S = \Sigma \cap \mathcal{F}$, $\tilde{S} = \Sigma \cap \tilde{\mathcal{F}}$, $S' = \{v'_h \mid v'_h = F(v_h),\ v_h \in S\}$ and $\tilde{S}' = \{v'_h \mid v'_h = F(v_h),\ v_h \in \tilde{S}\}$. we shall obtain finite element approximations of u (and $\tilde{u}$) in the classes S and S' ($\tilde{S}$ and $\tilde{S}'$ resp.), and error estimates of them for u (and $\tilde{u}$ resp.). By §3.2 we see that the finite element approximations of u and $\tilde{u}$ express their singular properties exactly near each critical point of $\partial\Omega$.

3.4. Finite element approximations ω_h and $\tilde{\omega}_h$. By ω_h and $\tilde{\omega}_h$ we denote the functions of S and $\tilde{S}$ resp. which satisfy the conditions

$$D(\omega_h) = \min_{v_h\in S} D(v_h) \quad \text{and} \quad D(\tilde{\omega}_h) = \min_{v_h\in\tilde{S}} D(v_h) \quad \text{resp.} \tag{3.2}$$

We call ω_h and $\tilde{\omega}_h$ the *finite element approximations of* u and $\tilde{u}$ *in* S and $\tilde{S}$ respectively. The following lemma follows from the minimal properties (3.2) by standard arguments.

Lemma 3.2. (i) *The following inequalities hold:*

$$D(v_h - \omega_h) = D(v_h) - D(\omega_h) \qquad \textit{for each } v_h \in S \quad \textit{and}$$
$$D(v_h - \tilde{\omega}_h) = D(v_h) - D(\tilde{\omega}_h) \qquad \textit{for each } v_h \in \tilde{S}.$$

(ii) *The functions* ω_h *and* $\tilde{\omega}_h$ *have the minimal properties*

$$D(\omega_h - u) \;=\; \min_{v_h \in S} D(v_h - u) \quad \textit{and} \quad D(\tilde{\omega}_h - \tilde{u}) \;=\; \min_{v_h \in \tilde{S}} D(v_h - \tilde{u}),$$

where the minimums are attained if and only if $v_h = \omega_h$ *and* $v_h = \tilde{\omega}_h + \mathrm{const.}$ *respectively.*

3.5. Finite element approximations u'_h and $\tilde{u}'_h$. By u'_h and $\tilde{u}'_h$ we denote the functions of S' and $\tilde{S}'$ respectively which satisfy the conditions

$$D(u'_h) \;=\; \min_{v'_h \in S'} D(v'_h) \quad \text{and} \quad D(\tilde{u}'_h) \;=\; \min_{v'_h \in \tilde{S}'} D(v'_h) \qquad \text{resp.} \tag{3.3}$$

We call u'_h and $\tilde{u}'_h$ the *finite element approximations of* u and $\tilde{u}$ *in* S' and $\tilde{S}'$ respectively. The functons u'_h and $\tilde{u}'_h$ can be obtained by solving some systems of linear equations.

3.6. Lemma of Bramble and Zlámal. The following lemma is due to J. H. Bramble and M. Zlámal (cf. Theorem 2 of [3]).

Lemma 3.3. *Let* Δ *be a closed triangle on the* z*-plane* $(z = x + iy)$ *with* $d(\Delta) \leq h$ *and let* v *be a function of the class* C^2 *defined on* Δ *such that* $v = 0$ *at each vertex of* Δ*. Then, the inequality*

$$\iint_\Delta \left(\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right) dxdy$$

$$\leq \frac{B}{\sin^2 \theta} h^2 \iint_\Delta \left(\left(\frac{\partial^2 v}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 v}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 v}{\partial y^2} \right)^2 \right) dxdy$$

holds, where B *is an absolute constant and* θ *is the smallest interior angle of the triangle* Δ.

3.7. Approximations by ω_h and $\tilde{\omega}_h$.

Theorem 3.1. *Let* u *and* $\tilde{u}$ *be the harmonic solutions in* $\mathcal{F}$ *and* $\tilde{\mathcal{F}}$ *respectively defined in* §3.3, *and let* ω_h *and* $\tilde{\omega}_h$ *be the finite element approximations of* u *and* $\tilde{u}$ *in* S *and* $\tilde{S}$ *respectively. Then,*

$$\begin{aligned} &D(\omega_h - u) \\ &\leq \frac{h^2}{\sin^2 \theta} \left(B \sum_{j=1}^{m} \iint_{\varphi_j(|K'_j|)} \left(\left(\frac{\partial^2 u}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 u}{\partial y^2} \right)^2 \right) dxdy \right. \\ &\qquad \left. + C \sum_{j=1}^{m} \max_{\varphi_j(R_j)} \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right) \right), \end{aligned} \tag{3.4}$$

where B *is an absolute constant,* C *is a constant dependent only on transformations of local parameters,* θ *is the smallest value of interior angles of all triangles* $\varphi_j(s)$ $(s \in K'_j;\ j = 1, \cdots, m)$, *and* R_j $(j = 1, \cdots, m)$ *are the closed subsets of* $U_j \cap \overline{\Omega}$ *defined in* (i′) *of* §1.2. *Further, when we replace* u *and* ω_h *by* $\tilde{u}$ *and* $\tilde{\omega}_h$ *respectively, the inequality* (3.4) *also holds.*

Proof. First, by (ii) of Lemma 3.2,

$$D(\omega_h - u) \;\leq\; D(\widehat{u} - u), \tag{3.5}$$

where $\widehat{u}$ is the finite element interpolation of u (see §2.4). Hence it is sufficient to estimate $D(\widehat{u} - u)$. We can write $D(\widehat{u} - u)$ as

$$D(\widehat{u} - u) \;=\; \sum_{j=1}^{m} \sum_{s \in K_j} D_s(\widehat{u} - u). \tag{3.6}$$

Here we note that $u \circ \varphi_j^{-1}$ $(j = 1, \cdots, m)$ is of the class C^2 on $\varphi_j(U_j \cap \overline{\Omega})$ (see §3.2). If s is a natural simplex of K_j which is not an adjoint of any minor simplex, then, by Lemma 3.3,

$$\begin{aligned} (3.7) \qquad & D_s(\widehat{u}-u) \\ & \leq \frac{B}{\sin^2\theta} h^2 \iint_{\varphi_j(s)} \left(\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 \right) dxdy. \end{aligned}$$

For simplicity, we denote the right hand side of (3.7) by $I[\varphi_j(s)]$.

If (s, s', ℓ) is a triple for a major simplex s of K_j, then, by Lemma 3.3

$$(3.8) \qquad D_s(\widehat{u}-u) \leq I[\varphi_j(\natural s)] + D_\ell(\widehat{u}-u) \leq I[\varphi_j(\natural s)] + 2\,D_\ell(\widehat{u}) + 2\,D_\ell(u).$$

Here by (i) of Lemma 2.2

$$(3.9) \qquad D_\ell(u) \leq A(\varphi_j(\ell)) \max_{\varphi_j(\ell)} \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right) \qquad \text{and}$$

$$(3.10) \qquad D_\ell(\widehat{u}) \leq A(\varphi_j(\ell))\, |(\nabla \widehat{u}')_{\natural s}|^2\, (1+\kappa h).$$

Further, by using the mean value theorem, we can prove

$$(3.11) \qquad |(\nabla \widehat{u}')_{\natural s}|^2 \leq \frac{8}{\sin^2\theta} \max_{\varphi_j(\natural s)} \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right).$$

If (s, s', ℓ) is a triple for a minor simplex s of K_j, then by Lemma 3.3

$$(3.12) \qquad D_s(\widehat{u}-u) \leq I[\varphi_j(\natural s)] \qquad \text{and}$$

$$\begin{aligned} (3.13) \qquad D_{s'}(\widehat{u}-u) & \leq I[\varphi_k(s')] + D_\ell(\widehat{u}-u) \\ & \leq I[\varphi_k(s')] + 2\,D_\ell(\widehat{u}) + 2\,D_\ell(u). \end{aligned}$$

Here for $D_\ell(u)$ the inequality (3.9) holds. We denote the branches of the function $\widehat{u}' = F(\widehat{u})$ on $\natural s$ and s' by $\widehat{u}'_s$ and $\widehat{u}'_{s'}$ respectively, and we introduce a function H on ℓ by $H = \widehat{u}'_s + \widehat{u}'_{s'} - \widehat{u}$. Then

$$(3.14) \qquad D_\ell(\widehat{u}) \leq 3\Big(D_\ell(\widehat{u}'_s) + D_\ell(\widehat{u}'_{s'}) + D_\ell(H)\Big).$$

Here, by (3.11) and (ii) of Lemma 2.2

$$\begin{aligned} (3.15) \qquad D_\ell(\widehat{u}'_s) & = A(\varphi_j(\ell))\, |\nabla \widehat{u}'_s|^2 \\ & \leq \frac{8}{\sin^2\theta} A(\varphi_j(\ell)) \cdot \max_{\varphi_j(\natural s)} \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right), \end{aligned}$$

$$\begin{aligned} (3.16) \qquad D_\ell(\widehat{u}'_{s'}) & = A(\varphi_k(\ell))\, |\nabla \widehat{u}'_{s'}|^2 \\ & \leq \frac{8}{\sin^2\theta} A(\varphi_k(\ell)) \cdot \max_{\varphi_k(s')} \left(\left(\frac{\partial u}{\partial \xi}\right)^2 + \left(\frac{\partial u}{\partial \eta}\right)^2 \right) \qquad \text{and} \end{aligned}$$

$$(3.17) \qquad D_\ell(H) \leq \frac{8}{\sin^2\theta} A(\varphi_j(\ell)) \cdot \max_{\varphi_j(\natural s)} \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right) \cdot (1+\kappa h).$$

By (3.5) $\sim$ (3.17), Lemma 1.1 and (ii$'$) of §1.2, the estimate (3.4) is obtained. By a similar method the remained part is proved.

3.8. Approximations by u'_h and $\tilde{u}'_h$.

Theorem 3.2. *Let u and $\tilde{u}$ be the harmonic solutions in $\mathcal{F}$ and $\tilde{\mathcal{F}}$ respectively defined in §3.3, let u'_h and $\tilde{u}'_h$ be the finite element approximations of u and $\tilde{u}$ in the classes S' and $\tilde{S}'$ respectively, and let $u_h = F^{-1}(u'_h)$ and $\tilde{u}_h = F^{-1}(\tilde{u}'_h)$.*

(i) *The estimate*

$$(3.18)\qquad D(u_h - u) \le \frac{h^2}{\sin^2\theta}\left(B'\sum_{j=1}^{m}\iint_{\varphi_j(|K'_j|)}\left(\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2\right)dxdy + C'\sum_{j=1}^{m}\max_{\varphi_j(R_j)}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right)\right)$$

holds, where B' and C' are constants dependent only on transformations of local parameters, and other notations are the same as in Theorem 3.1. *Further, when we replace u and u_h by $\tilde{u}$ and $\tilde{u}_h$ respectively, the estimate* (3.18) *also holds.*

(ii) *The following estimate holds with $\varepsilon(u'_h)$ defined by* (2.8)*:*

$$(3.19)\qquad D(u) \le D(u'_h) + \varepsilon(u'_h).$$

Further, when we replace u and u'_h by $\tilde{u}$ and $\tilde{u}'_h$ respectively, the estimate (3.19) *also holds.*

By (iii) *of Lemma* 2.2, *$\varepsilon(u'_h)$ is a quantity of $O(h^2)$ which can be numerically calculated if u'_h is obtained.*

Proof. (i) We note that

$$(3.20)\qquad D(u_h - u) \le 2D(\omega_h - u) + 2D(u_h - \omega_h).$$

Here, by (i) of Lemma 3.2, (iii) of Lemma 2.2, (3.3), and the definition of ω_h and ω'_h

$$D(u_h - \omega_h) = D(u_h) - D(\omega_h) \le \left(D(u'_h) + \varepsilon(u'_h)\right) - D(\omega_h)$$

$$\le D(\omega'_h) - D(\omega_h) + \varepsilon(u'_h) \le \sum_{\flat\ell\in K}\left(D_{\flat\ell}(\omega'_{hs}) + D_{\flat\ell}(\omega'_{hs'})\right) + \varepsilon(u'_h).$$

By (2.8) the last inequality implies

$$(3.21)\qquad D(u_h - \omega_h) \le \sum_{\sharp\ell\in K}\varepsilon_{\sharp\ell}(u'_h) + \sum_{\flat\ell\in K}\left(3\varepsilon_{\flat\ell}(u'_h) + 2\left(D_{\flat\ell}(u'_{hs}) + D_{\flat\ell}(u'_{hs'})\right) + \left(D_{\flat\ell}(\omega'_{hs}) + D_{\flat\ell}(\omega'_{hs'})\right)\right).$$

We shall obtain an estimate for each term of the right hand side of (3.21). First, we have the estimate

$$(3.22)\qquad D_{\flat\ell}(u'_{hs}) = A(\varphi_j(\flat\ell))\,|\nabla u'_{hs}|^2 = \frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))}\,D_s(u_h)$$

$$\le 2\,\frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))}\left(D_s(u_h - u) + D_s(u)\right)$$

$$\le 2\,\frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))}\,D_s(u_h - u) + 2A(\varphi_j(\flat\ell))\cdot\max_{\varphi_j(s)}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right).$$

Since the inequality $A(\varphi_j(\natural s)) > (h_1^2/4)\sin\theta$ $(h_1 = d(\varphi_j(\natural s)))$ holds, by Lemma 1.1

$$(3.23)\qquad \frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))} = \frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(\natural s)) - A(\varphi_j(\flat\ell))} \le \frac{h}{2\sin\theta}\left(\left|\frac{\psi''(\zeta_1)}{\psi'(\zeta_1)^2}\right| + \kappa h\right)$$

with the notations in Lemma 1.1. (3.22) and (3.23) imply the estimate

$$(3.24)\qquad \sum_{\flat\ell\in K} D_{\flat\ell}(u'_{hs})$$

$$\le \frac{C_1 h}{\sin\theta}\sum_{j=1}^{m}\sum_{\flat\ell\in K_j} D_s(u_h - u) + 2\sum_{j=1}^{m}\sum_{\flat\ell\in K_j} A(\varphi_j(\flat\ell))\cdot\max_{\varphi_j(s)}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right),$$

where C_1 is a constant dependent only on the transformations of local parameters. Next, by (i) of Lemma 2.2 we have the inequality

$$(3.25)\qquad \varepsilon_{\sharp\ell}(u'_h) \le A(\varphi_j(\sharp\ell))\,|(\nabla u'_h)_{\natural s}|^2\,(1+\kappa h),$$

and thus a similar estimate as (3.24) for $\sum_{\sharp\ell\in K}\varepsilon_{\sharp\ell}(u'_h)$ is obtained. Since similar estimates as (3.24) for other terms of the right hand side of (3.21) are obtained, from (3.21) it follows that

$$(3.26)\qquad D(u_h - \omega_h) \le \frac{B_1 h}{\sin\theta}\Big(D(u_h - u) + D(\omega_h - u)\Big)$$

$$+B_2\sum_{j=1}^{m}\sum_{\ell\in K_j}\left(A(\varphi_j(\ell))\cdot\max_{\varphi_j(s)}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) + A(\varphi_k(\ell))\cdot\max_{\varphi_k(s')}\left(\left(\frac{\partial u}{\partial \xi}\right)^2 + \left(\frac{\partial u}{\partial \eta}\right)^2\right)\right),$$

where B_1 and B_2 are constants dependent only on the transformations of local parameters. (3.20), (3.26), Theorem 3.1, Lemma 1.1 and (ii$'$) of §1.2 yield the estimate (3.18).

By a similar method the remained part of (i) is proved.

(ii) By Lemma 3.1 and (iii) of Lemma 2.2, we have the estimate

$$(3.27)\qquad D(u) \le D(u_h) \le D(u'_h) + \varepsilon(u'_h).$$

By a similar method the remained part of (ii) is proved.

§4. Application

4.1. Estimation of modulus. Let $M(\Omega)$ be the modulus of a ring domain Ω, let u and $\tilde u$ be the harmonic solutions in the classes $\mathcal{F}$ and $\tilde{\mathcal{F}}$ respectively, and let u'_h and $\tilde u'_h$ be the finite element approximations of u and $\tilde u$ in the classes S' and $\tilde S'$ respectively. Then by Lemma 3.1 and (ii) of Theorem 3.2, we have upper and lower bounds for the modulus $M(\Omega)$:

$$(4.1)\qquad \frac{2\pi}{D(u'_h) + \varepsilon(u'_h)} \le M(\Omega) \le 2\pi(D(\tilde u'_h) + \varepsilon(\tilde u'_h)),$$

where $\varepsilon(u'_h)$ and $\varepsilon(\tilde u'_h)$ are the numerically computable quantities defined by (2.8).

4.2. Circular domain with a slit. Let Ω be the ring domain on the z-plane defined by $\Omega = \{z \mid |z| < 1\} - \{z \mid -a \le x \le a,\ y = 0\}$ $(z = x + iy;\ 0 < a < 1)$, and let C_0 and C_1 be the boundary components of Ω lying on $\{z \mid -a \le x \le a,\ y = 0\}$ and $\{z \mid |z| = 1\}$ $(z = x + iy)$ respectively (cf. Fig. 6). Then for a given value of a the exact value of the modulus $M(\Omega)$ of the domain Ω can be numerically calculated by using of the elliptic integral (cf. p.62 of [16]).

In the cases where a are sufficiently near to 1 (circular domain with a long slit) and sufficiently near to 0 (circular domain with a short slit), we aim to obtain good upper and lower approximate values of $M(\Omega)$.

4.3. Numerical examples of circular domains with a long slit. Now we shall treat of the cases

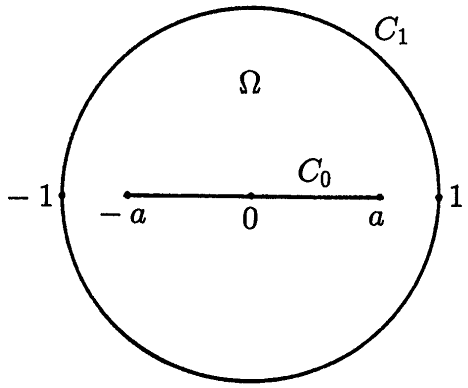

Fig. 6 A circular domain with a slit

where a is sufficiently near to 1. First, as in Fig. 7 we determine the collection Φ of local parameters $\zeta_1 = \varphi_1(z)$, $\zeta_2 = \varphi_2(z)$ and $\zeta_3 = \varphi_3(z)$, and parametric regions G_1, G_2 and G_3 of the closed region $\overline{D} = \{z \mid \mathrm{Re}\, z \geq 0,\ \mathrm{Im}\, z \geq 0\} \cap \overline{\Omega}$ satisfying the conditions of §1.1, where $\overline{\Omega}$ and $\overline{D}$ are regarded as bordered Riemann surfaces. In Fig. 7, $G_1^* = \{\zeta_1 \mid 0 \leq \mathrm{Re}\,\zeta_1 \leq a_0,\ 0 \leq \mathrm{Im}\,\zeta_1 \leq c_1\pi/2\}$, $G_1 = \varphi_1^{-1}(G_1^*)$, $G_2^* = \{\zeta_2 \mid b \leq \mathrm{Re}\,\zeta_2 \leq 0,\ 0 \leq \mathrm{Im}\,\zeta_2 \leq \pi\} \cap \overline{\varphi_2(D - G_1)}$, $G_2 = \varphi_2^{-1}(G_2^*)$, $G_3 = \overline{D - G_1 \cup G_2}$ and $G_3^* = \varphi_3(G_3)$, where a_0 and b are so chosen that $a_0 = \varphi_1(2a-1)$ and $b = -\log 2$. In Fig. 7, though the constant c $(2a - 1 < c < a)$ may be optionally chosen, it is determined so that $\lambda|l'(z_0)| = (1-a)|l'(1)|$, where $l(z) \equiv (z-c)/(1-cz)$ and λ is length of the arc between $2a-1$ and z_0.

Next, as in Fig. 8 we determine the triangulation of $\overline{D}$ associated to the present Φ satisfying the conditions of §1.2. We construct the triangulation K_1 of G_1 whose each 2-simplex is natural and namely in Fig. 8, K_1^* is an ordinary triangulation. The triangulation K_2 of G_2 is so constructed that each 2-simplex s of K_2 is natural or minor according as $|s| \cap |K_1| = \emptyset$ or $|s| \cap |K_1| \neq \emptyset$ respectively (see K_2^* in Fig. 8), where if some intersection is a point then it is interpreted to be vacuous. The triangulation K_3 of G_3 is so constructed that each 2-simplex s of K_3 is natural, minor or major according as $|s| \cap |K_1 + K_2| = \emptyset$, $|s| \cap |K_1| \neq \emptyset$ or $|s| \cap |K_2| \neq \emptyset$ respectively (see K_3^* in Fig. 8). The triangulation K of $\overline{\Omega}$ is obtained by iteration of reflections of the triangulation of $\overline{D}$. We see that the triangulation K conforms to the definition in §1.2.

Table 1 shows the computational results of the case $a = 0.9$ (then $c = 0.8567$, $c_1 = 0.9748$ and $c_3 = 2.5979$) and those for the normal subdivision K^1. Table 2 shows those of the case $a = 0.999$ (then $c = 0.9986$, $c_1 = 0.9993$ and $c_3 = 2.6286$). It can be said that our results are close to the exact moduli.

4.4. Numerical examples of circular domains with a short slit. Next we treat of the cases where a is sufficiently near to 0. We determine the collection Φ of local parameters and parametric regions of the closed region $\overline{D} = \{z \mid \mathrm{Re}\, z \geq 0,\ \mathrm{Im}\, z \geq 0\} \cap \overline{\Omega}$ as in Fig. 9, and construct a triangulation of each parametric region as in Fig. 10. The remained arguments are similar to §4.3.

Here we know that by using of the mapping $w = \log(z/a + (z^2/a^2 - 1)^{1/2})$ the quantities: $M_1 \equiv \log((1/a)(1 + (1-a^2)^{1/2}))$ and $M_2 \equiv \log((1/a)(1 + (1+a^2)^{1/2}))$ give the lower and upper bounds of modulus $M(\Omega)$ respectively: $M_1 < M(\Omega) < M_2$. In fact, the case of $a = 0.1$: $M_2 - M(\Omega) = 0.0025$ and $M_1 - M(\Omega) = -0.0024$, and the case of $a = 0.001$: $M_2 - M(\Omega) = 0.00000025$ and $M_1 - M(\Omega) = -0.00000024$. Our aim is to show that our finite element results give much better estimates than the above ones. Table 3 shows the computational results of the case $a = 0.1$ (then $c_1 = 1.0004$) and those for the normal subdivision K^1. Table 4 shows those of the case $a = 0.001$ (then $c_1 = 1.0000$).

4.5. Square domain with a slit. Let Ω be the ring domain on the z-plane defined by $\Omega = \{z \mid |x| <$

$1,\ |y| < 1\} - \{z \mid -a \leq x \leq a,\ y = 0\}$ $(z = x + iy;\ 0 < a < 1)$, and let C_0 and C_1 be the boundary components of Ω lying on $\{z \mid -a \leq x \leq a,\ y = 0\}$ and $\{z \mid |x| = 1,\ |y| \leq 1\} \cup \{z \mid |x| \leq 1,\ |y| = 1\}$ $(z = x + iy)$ respectively (cf. Fig. 11). Then for a given value of a the exact value of the mudulus $M(\Omega)$ of the domain Ω can be numerically calculated by using the elliptic integral (cf. p.63 of [16]).

4.6. Numerical examples of square domains with a long slit. Now we treat of the cases where a of the domain Ω is sufficiently near to 1. We determine the collection Φ of local parameters and parametric regions of the closed region $\overline{D} = \{z \mid \operatorname{Re} z \geq 0,\ \operatorname{Im} z \geq 0\} \cap \overline{\Omega}$ as in Fig. 12, and construst the triangulation of each parametric region as in Fig. 13. The remained arguments are similar to §4.3.

Table 5 shows the computational results of the case $a = 0.9$ (then $c = 0.8711$, $c_2 = 0.5867$, $c_3 = 0.6544$, $c_4 = 1.5996$ and $c_5 = 0.7081$) and those for the normal subdivision K^1. Table 6 shows those of the case $a = 0.999$ (then $c = 0.9986$, $c_2 = 0.5867$, $c_3 = 0.5805$, $c_4 = 1.5378$ and $c_5 = 0.5907$).

4.7. Numerical examples of square domains with a short slit. Next we treat of the cases where a is sufficiently near to 0. We determine the collection Φ of local parameters and parametric regions of the closed region $\overline{D} = \{z \mid \operatorname{Re} z \geq 0,\ \operatorname{Im} z \geq 0\} \cap \overline{\Omega}$ as in Fig. 14, and construst the triangulation of each parametric region as in Fig. 15. The remained arguments are similar to the case of §4.3.

Table 7 shows the computational results of the case $a = 0.1$ (then $c_1 = 0.6126$ and $c_4 = 0.6125$) and those for the normal subdivision K^1. Table 8 shows those of the case $a = 0.001$ (then $c_1 = 0.6127$ and $c_4 = 0.6127$).

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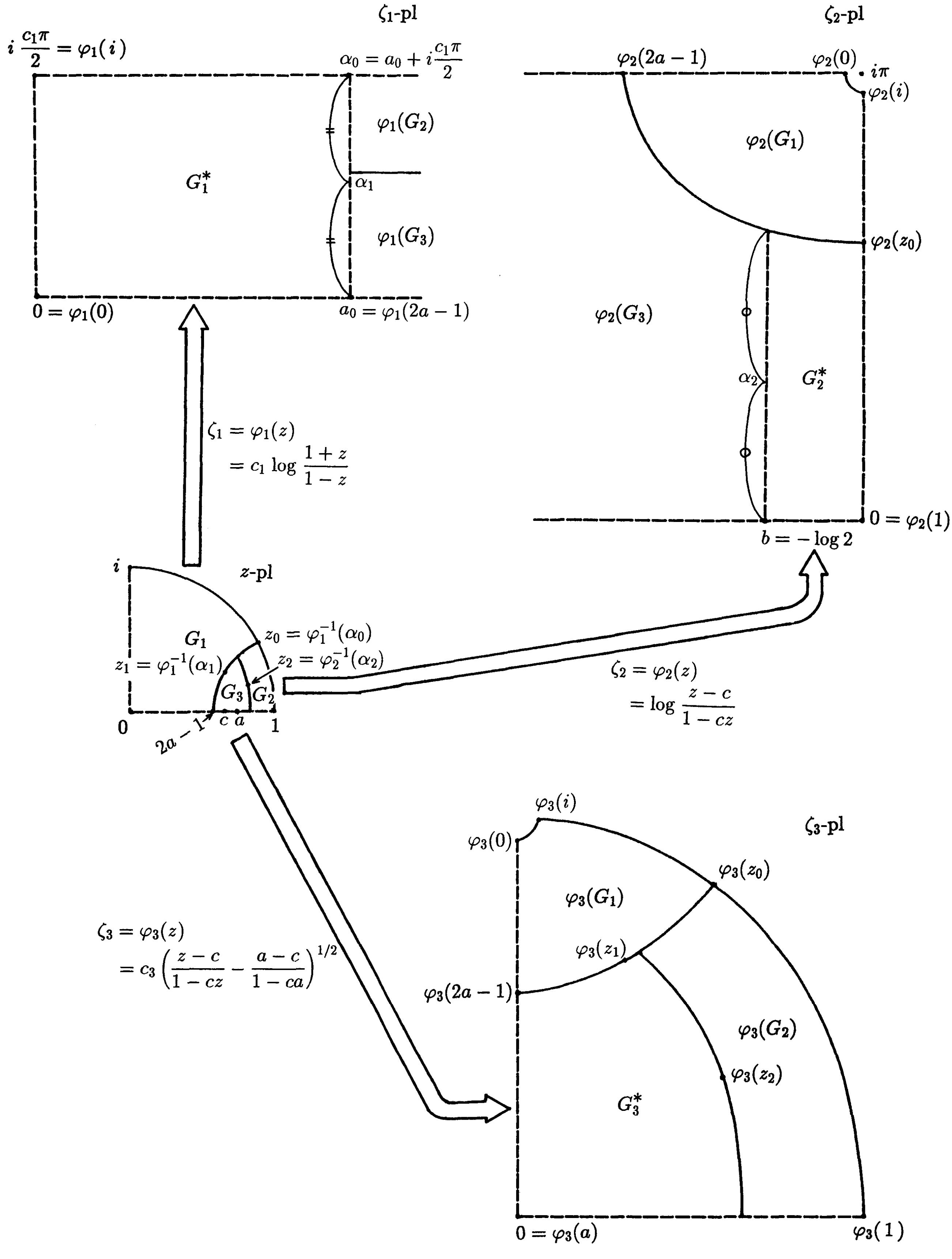

Fig. 7 Local parameters and parametric regions of a circular domain with a long slit.
c_1 and c_3 are so determined that $|d\zeta_1/dz| = |d\zeta_3/dz|$ at $z = z_1$ and $|d\zeta_2/dz| = |d\zeta_3/dz|$ at $z = z_2$.
— — — $\cdots$ segment or half straight line being parallel to real axis or imaginary axis.

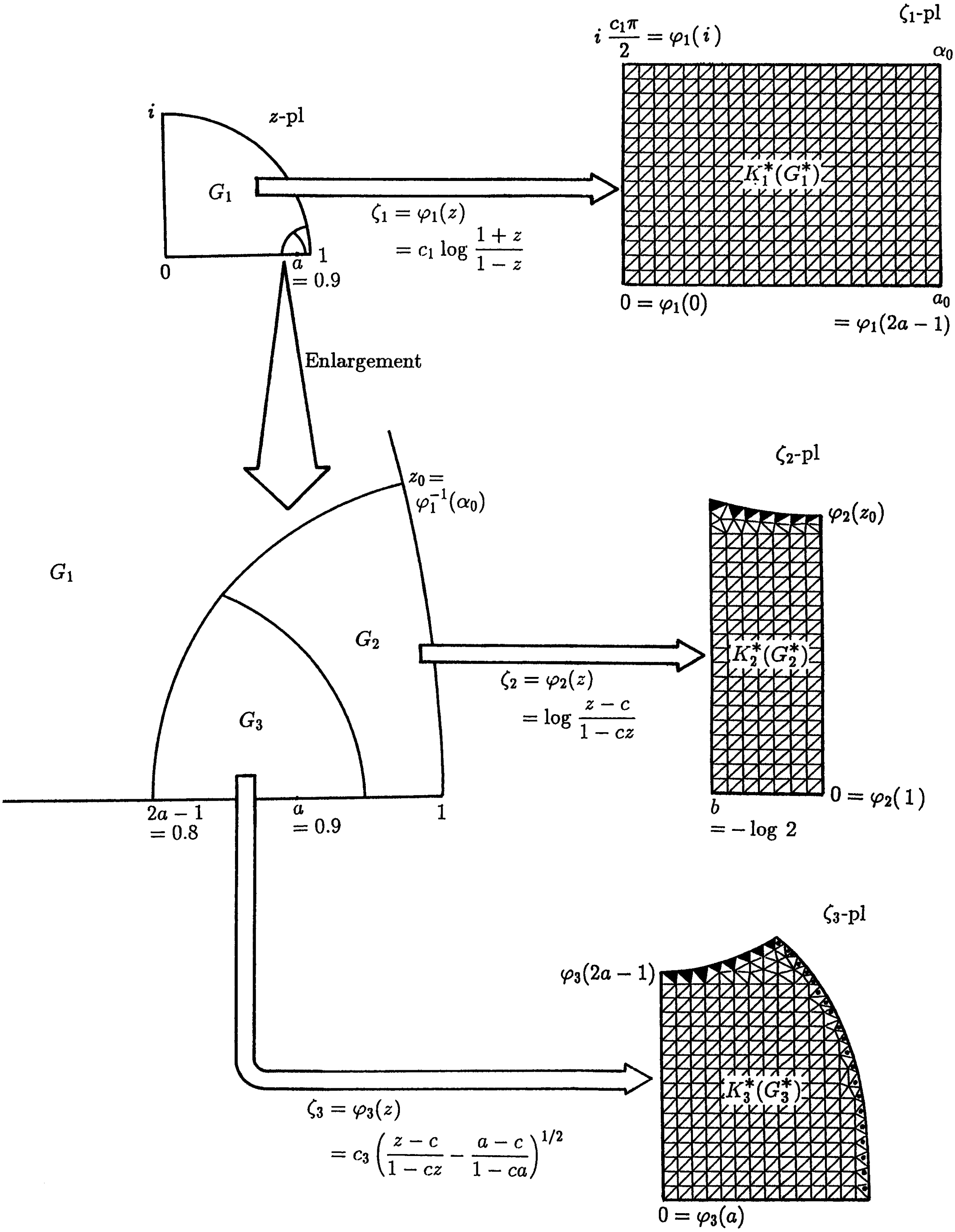

Fig. 8 Triangulation of a circular domain with a long slit ($a = 0.9$).
$\triangle$ $\cdots$ The local map $\varphi_j(s)$ of a major simplex s;
$\blacktriangle$ $\cdots$ The local map $\varphi_j(s)$ of a minor simplex s.

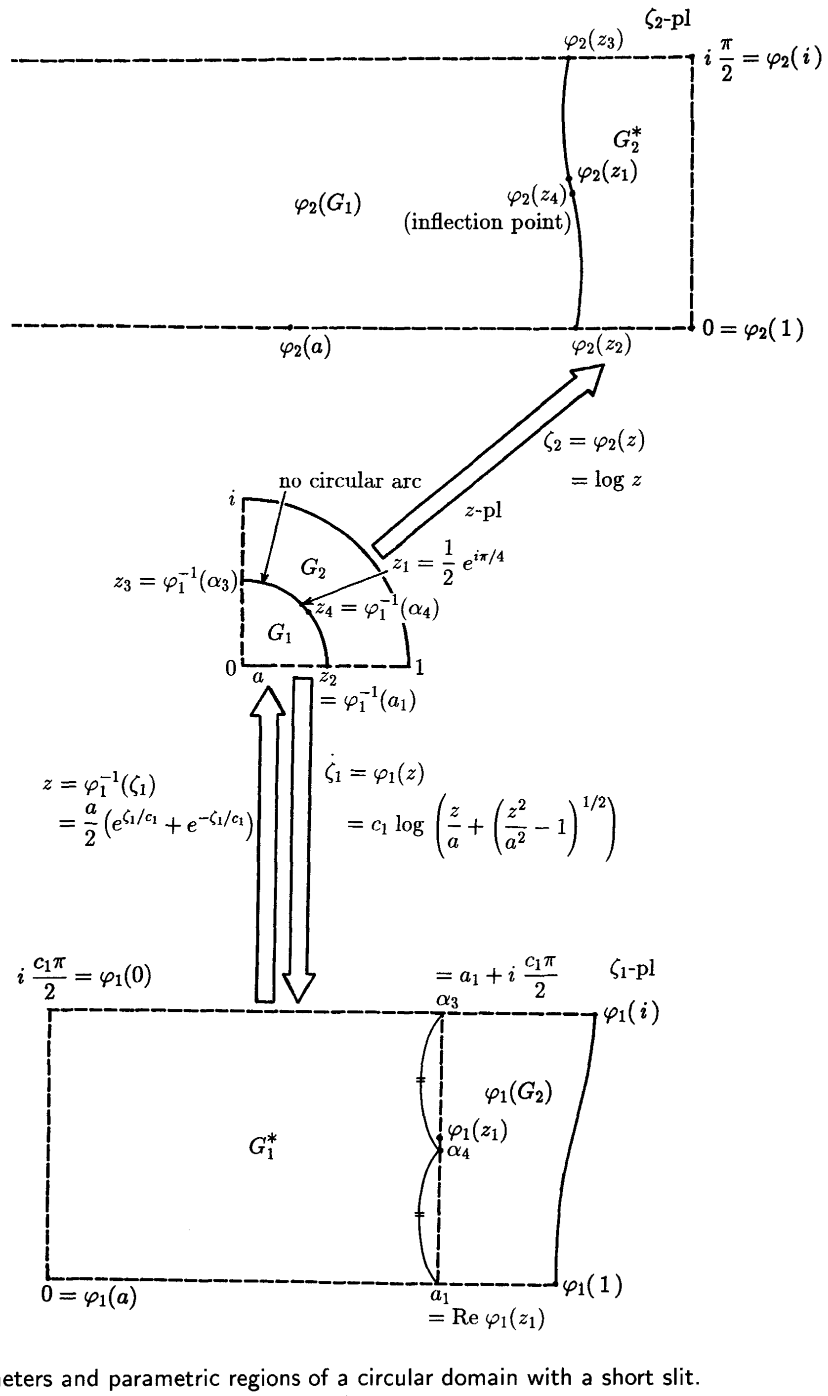

Fig. 9 Local parameters and parametric regions of a circular domain with a short slit.
c_1 is so determined that $|d\zeta_1/dz| = |d\zeta_2/dz|$ at $z = z_1$.
— — — ··· segment or half straight line being parallel to real axis or imaginary axis.

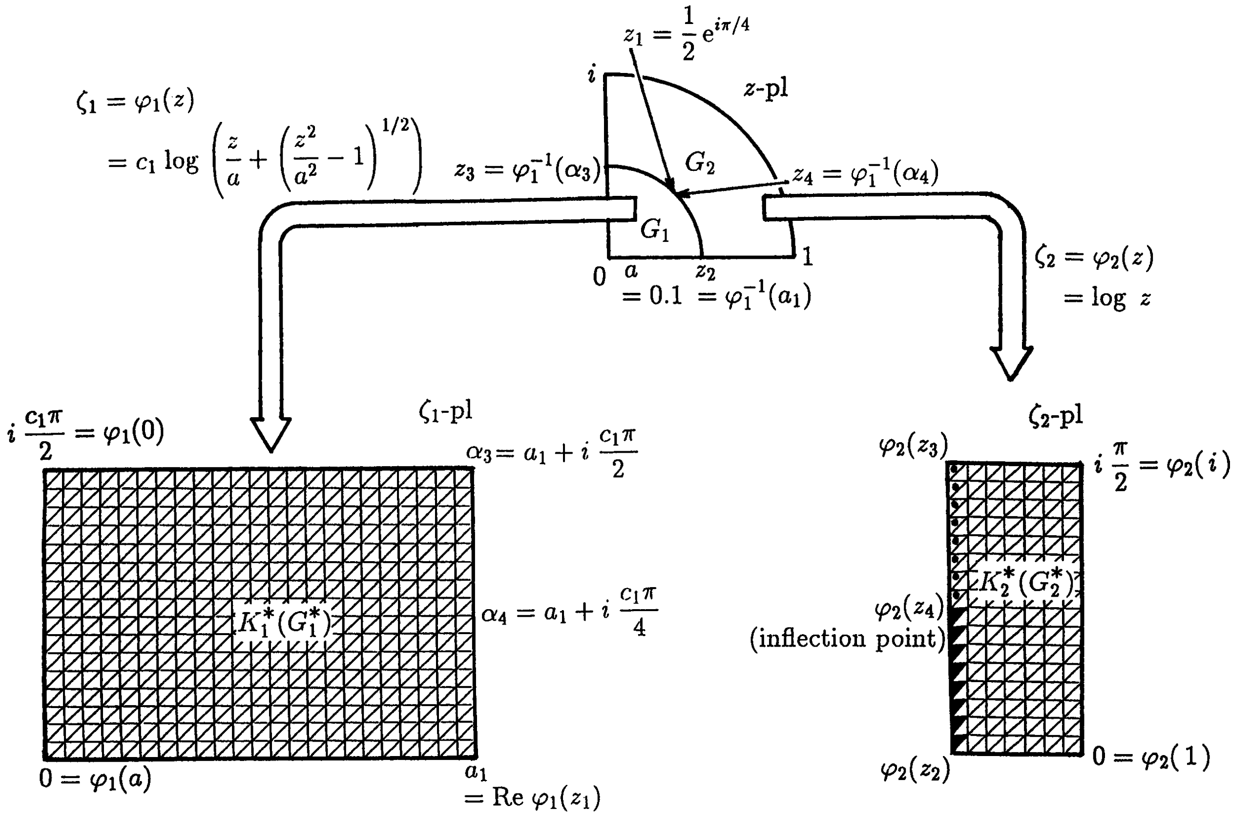

Fig. 10 Triangulation of a circular domain with a short slit ($a = 0.1$).
$\triangle$ $\cdots$ The local map $\varphi_2(s)$ of a major simplex s;
$\blacktriangle$ $\cdots$ The local map $\varphi_2(s)$ of a minor simplex s.

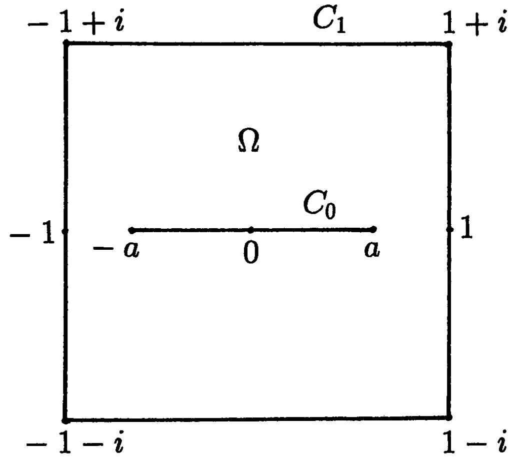

Fig. 11 A square domain with a slit

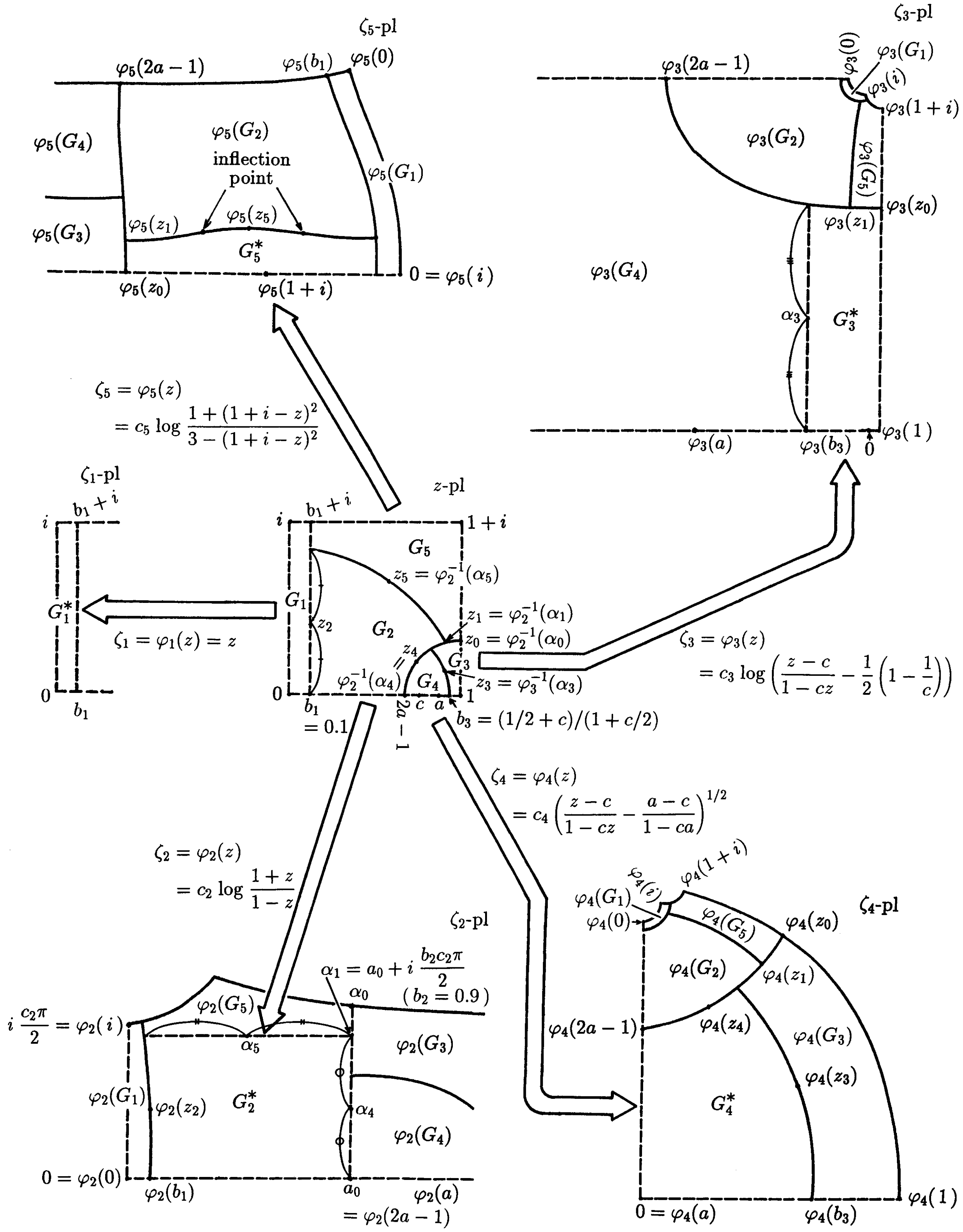

Fig. 12 Local parameters and parametric regions of a square domain with a long slit.
c_2, c_3, c_4 and c_5 are so determined that $|d\zeta_1/dz| = |d\zeta_2/dz|$ at $z = z_2$, $|d\zeta_4/dz| = |d\zeta_3/dz|$ at $z = z_3$, $|d\zeta_2/dz| = |d\zeta_4/dz|$ at $z = z_4$ and $|d\zeta_2/dz| = |d\zeta_5/dz|$ at $z = z_5$, and c is determined analogously to the case of §4.3.
— — — ··· segment or half straight line being parallel to real axis or imaginary axis.

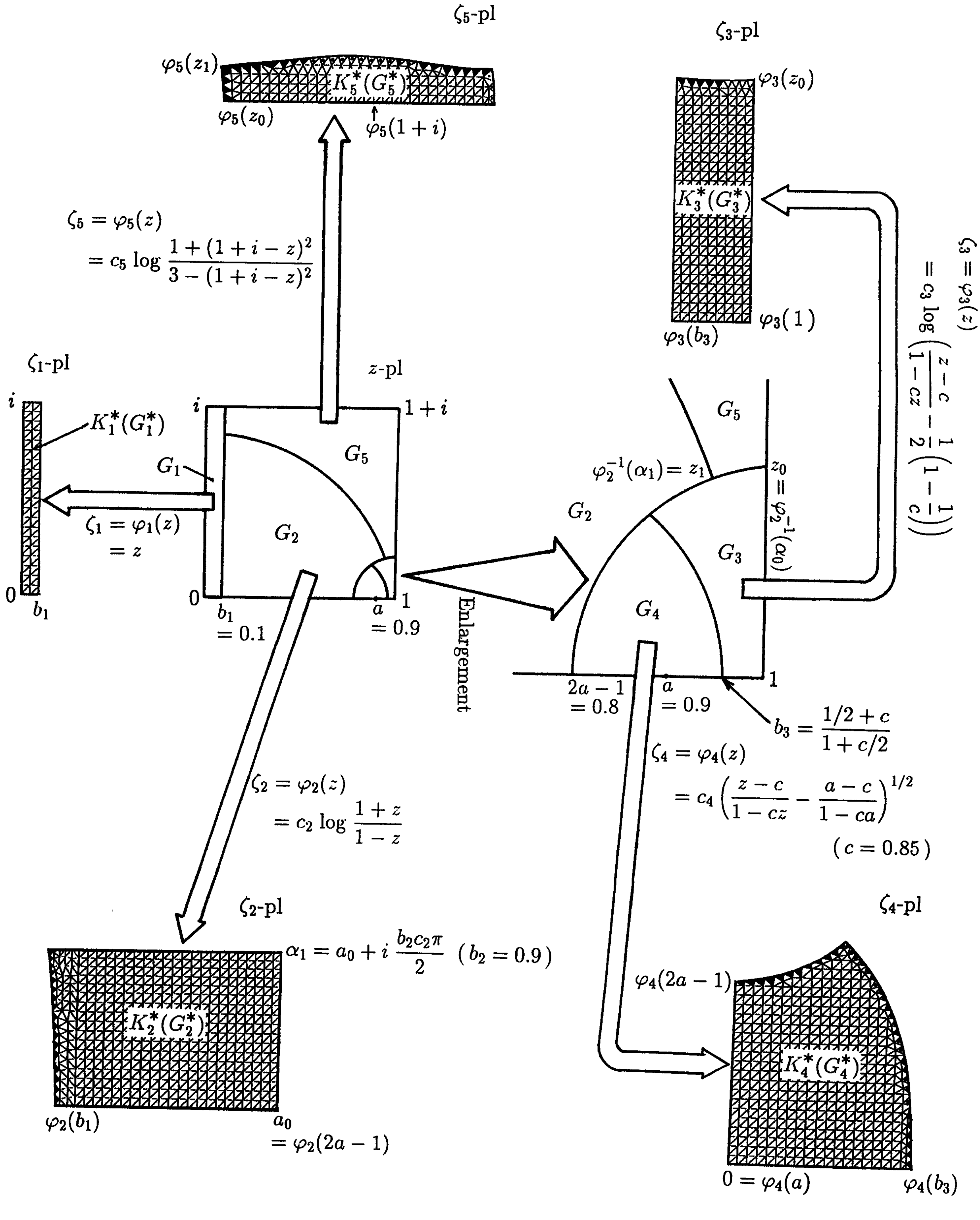

Fig. 13 Triangulation of a square domain with a long slit ($a = 0.9$).
$\triangle$ $\cdots$ The local map $\varphi_j(s)$ of a major simplex s;
$\blacktriangle$ $\cdots$ The local map $\varphi_j(s)$ of a minor simplex s.

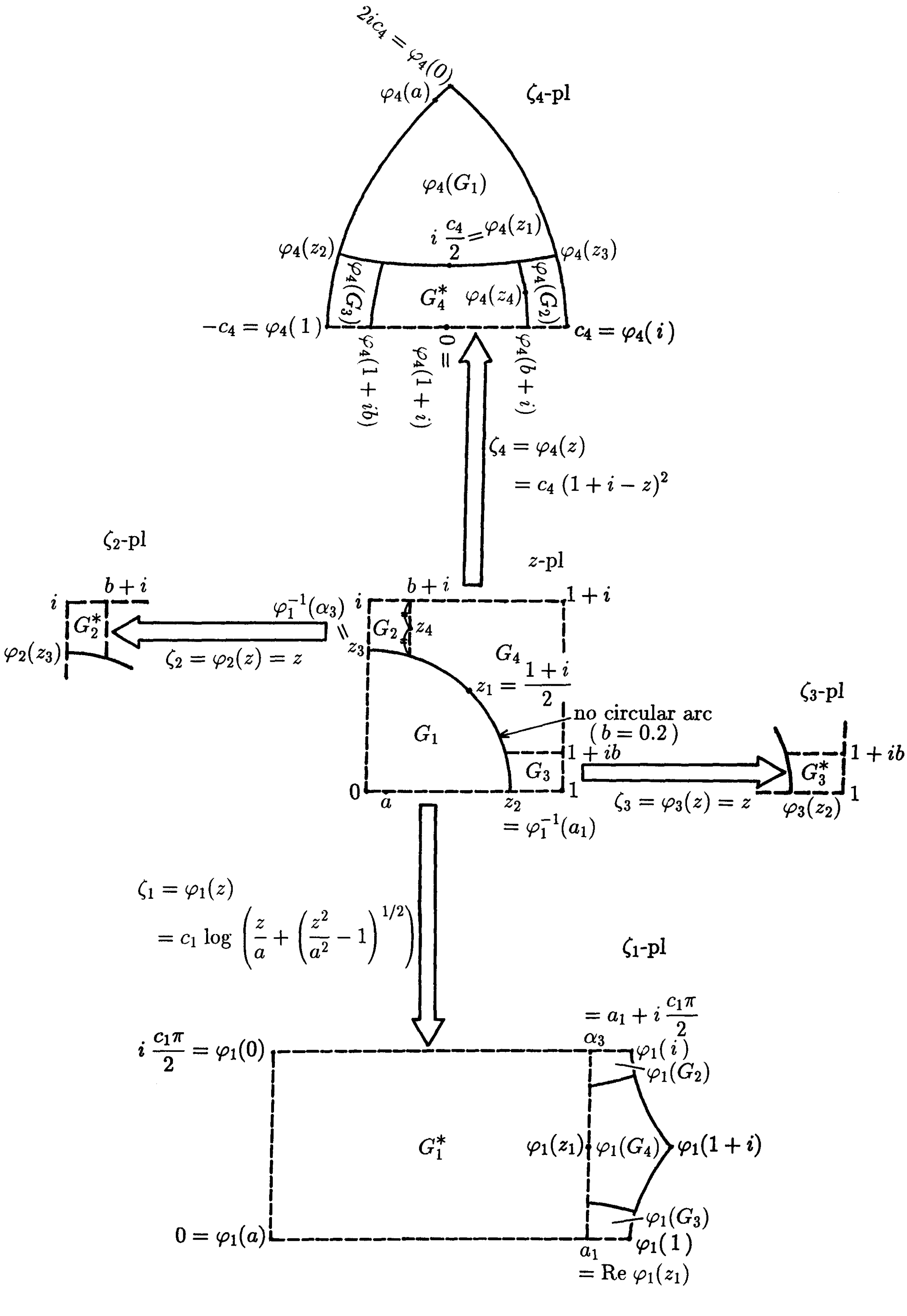

Fig. 14 Local parameters and parametric regions of a square domain with a short slit.
c_1 and c_4 are so determined that $|d\zeta_1/dz| = |d\zeta_4/dz|$ at $z = z_1$ and $|d\zeta_4/dz| = |d\zeta_2/dz|$ at $z = z_4$.
$---$ $\cdots$ segment or half straight line being parallel to real axis or imaginary axis.

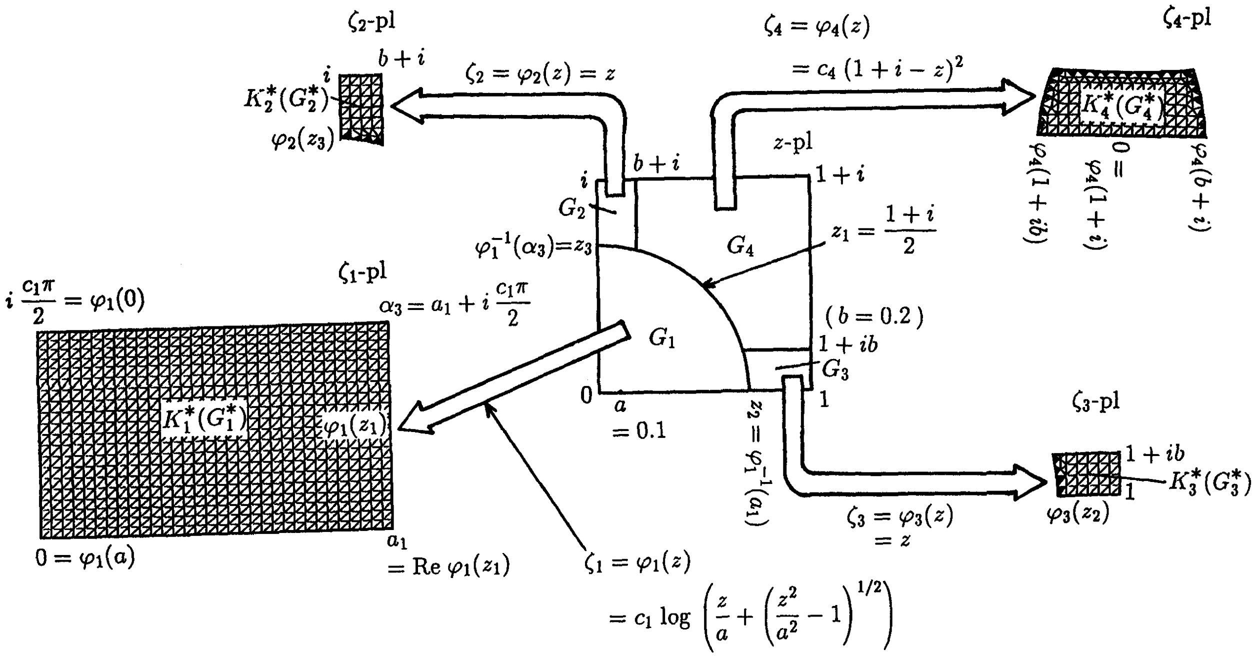

Fig. 15 Triangulation of a square domain with a short slit ($a = 0.1$).
$\triangle$ $\cdots$ The local map $\varphi_j(s)$ of a major simplex s;
$\blacktriangle$ $\cdots$ The local map $\varphi_j(s)$ of a minor simplex s.

Table 1 Modulus of a circular domain with a long slit ($a = 0.9$).

Exact values	$D(u) = 9.26303, \ D(\tilde{u}) = 0.1079559,$ $M(\Omega) = 2\pi/D(u) = 0.678307.$
Finite element approximations	Original triangulation $(h = 0.1 \times \sqrt{2},\ N(s) = 5456,\ N(q) = 2811)$ Upper bound $D(\tilde{u}'_h) = 0.1079447,\ \varepsilon(\tilde{u}'_h) = 0.0000409;$ $\overline{M} = 2\pi(D(\tilde{u}'_h) + \varepsilon(\tilde{u}'_h)) = 0.678494,$ $\overline{M} - M(\Omega) = 0.000186.$ Lower bound $D(u'_h) = 9.26564,\ \varepsilon(u'_h) = 0.00922;$ $\underline{M} = 2\pi/(D(u'_h) + \varepsilon(u'_h)) = 0.677442,$ $\underline{M} - M(\Omega) = -0.000865.$
	Normal subdivision $(h = 0.05 \times \sqrt{2},\ N(s) = 21824,\ N(q) = 11077)$ Upper bound $D(\tilde{u}'_h) = 0.1079532,\ \varepsilon(\tilde{u}'_h) = 0.0000103;$ $\overline{M} = 2\pi(D(\tilde{u}'_h) + \varepsilon(\tilde{u}'_h)) = 0.678354,$ $\overline{M} - M(\Omega) = 0.000047.$ Lower bound $D(u'_h) = 9.26369,\ \varepsilon(u'_h) = 0.00229;$ $\underline{M} = 2\pi/(D(u'_h) + \varepsilon(u'_h)) = 0.678092,$ $\underline{M} - M(\Omega) = -0.000216.$

$N(s)$: Number of 2-simplices, $N(q)$: Number of 0-simplices.

Table 2 Modulus of a circular domain with a long slit ($a = 0.999$).

Exact values	$D(u) = 21.11935,\ \ D(\tilde{u}) = 0.04734994,$ $M(\Omega) = 2\pi/D(u) = 0.2975085.$
Finite element approximations	Original triangulation $(h = 0.1\times\sqrt{2},\ N(s) = 11908,\ N(q) = 6133)$ Upper bound $D(\tilde{u}'_h) = 0.04734817,\ \varepsilon(\tilde{u}'_h) = 0.00000719;$ $\overline{M} = 2\pi(D(\tilde{u}'_h) + \varepsilon(\tilde{u}'_h)) = 0.2975425,$ $\overline{M} - M(\Omega) = 0.0000341.$ Lower bound $D(u'_h) = 21.12189,\ \varepsilon(u'_h) = 0.00820;$ $\underline{M} = 2\pi/(D(u'_h) + \varepsilon(u'_h)) = 0.2973572,$ $\underline{M} - M(\Omega) = -0.0001513.$
	Normal subdivision $(h = 0.05\times\sqrt{2},\ N(s) = 47632,\ N(q) = 24173)$ Upper bound $D(\tilde{u}'_h) = 0.04734950,\ \varepsilon(\tilde{u}'_h) = 0.00000181;$ $\overline{M} = 2\pi(D(\tilde{u}'_h) + \varepsilon(\tilde{u}'_h)) = 0.2975170,$ $\overline{M} - M(\Omega) = 0.0000086.$ Lower bound $D(u'_h) = 21.11999,\ \varepsilon(u'_h) = 0.00204;$ $\underline{M} = 2\pi/(D(u'_h) + \varepsilon(u'_h)) = 0.2974708,$ $\underline{M} - M(\Omega) = -0.0000377.$

$N(s)$: Number of 2-simplices, $N(q)$: Number of 0-simplices.

Table 3 Modulus of a circular domain with a short slit ($a = 0.1$).

Exact values	$D(u) = 2.09738753$, $D(\tilde{u}) = 0.476783610$, $M(\Omega) = 2\pi/D(u) = 2.9957198$.
Finite element approximations	Original triangulation ($h = 0.1 \times \sqrt{2}$, $N(s) = 3840$, $N(q) = 1953$) Upper bound $D(\tilde{u}'_h) = 0.476783758$, $\varepsilon(\tilde{u}'_h) = 0.00001307$; $\overline{M} = 2\pi(D(\tilde{u}'_h) + \varepsilon(\tilde{u}'_h)) = 2.9958029$, $\overline{M} - M(\Omega) = 0.0000831$. Lower bound $D(u'_h) = 2.09738824$, $\varepsilon(u'_h) = 0.0000144$; $\underline{M} = 2\pi/(D(u'_h) + \varepsilon(u'_h)) = 2.9956982$, $\underline{M} - M(\Omega) = -0.0000215$.
	Normal subdivision ($h = 0.05 \times \sqrt{2}$, $N(s) = 15360$, $N(q) = 7745$) Upper bound $D(\tilde{u}'_h) = 0.476783647$, $\varepsilon(\tilde{u}'_h) = 0.00000326$; $\overline{M} = 2\pi(D(\tilde{u}'_h) + \varepsilon(\tilde{u}'_h)) = 2.9957405$, $\overline{M} - M(\Omega) = 0.0000207$. Lower bound $D(u'_h) = 2.09738771$, $\varepsilon(u'_h) = 0.0000036$; $\underline{M} = 2\pi/(D(u'_h) + \varepsilon(u'_h)) = 2.9957144$, $\underline{M} - M(\Omega) = -0.0000054$.

$N(s)$: Number of 2-simplices, $N(q)$: Number of 0-simplices.

Table 4 Modulus of a circular domain with a short slit ($a = 0.001$).

Exact values	$D(u) = 0.82663675012588194$, $D(\tilde{u}) = 1.20972119839544751$, $M(\Omega) = 2\pi/D(u) = 7.60090245954$.
Finite element approximations	Original triangulation ($h = 0.1 \times \sqrt{2}$, $N(s) = 9728$, $N(q) = 4897$) Upper bound $D(\tilde{u}'_h) = 1.20972119839544898$, $\varepsilon(\tilde{u}'_h) = 0.000000001304$; $\overline{M} = 2\pi(D(\tilde{u}'_h) + \varepsilon(\tilde{u}'_h)) = 7.60090246774$, $\overline{M} - M(\Omega) = 0.00000000819$. Lower bound $D(u'_h) = 0.82663675012588301$, $\varepsilon(u'_h) = 0.000000000223$; $\underline{M} = 2\pi/(D(u'_h) + \varepsilon(u'_h)) = 7.60090245749$, $\underline{M} - M(\Omega) = -0.00000000205$.
	Normal subdivision ($h = 0.05 \times \sqrt{2}$, $N(s) = 38912$, $N(q) = 19521$) Upper bound $D(\tilde{u}'_h) = 1.20972119839544788$, $\varepsilon(\tilde{u}'_h) = 0.000000000326$; $\overline{M} = 2\pi(D(\tilde{u}'_h) + \varepsilon(\tilde{u}'_h)) = 7.60090246159$, $\overline{M} - M(\Omega) = 0.00000000205$. Lower bound $D(u'_h) = 0.82663675012588221$, $\varepsilon(u'_h) = 0.000000000056$; $\underline{M} = 2\pi/(D(u'_h) + \varepsilon(u'_h)) = 7.60090245903$, $\underline{M} - M(\Omega) = -0.00000000051$.

$N(s)$: Number of 2-simplices, $N(q)$: Number of 0-simplices.

Table 5 Modulus of a square domain with a long slit ($a = 0.9$).

Exact values	$D(u) = 8.69309$, $D(\tilde{u}) = 0.1150338$, $M(\Omega) = 2\pi/D(u) = 0.722778$.
Finite element approximations	Original triangulation $(h = 0.05 \times \sqrt{2}$, $N(s) = 11784$, $N(q) = 5109)$ Upper bound $D(\tilde{u}_h') = 0.1150267$, $\varepsilon(\tilde{u}_h') = 0.0000473$; $\overline{M} = 2\pi(D(\tilde{u}_h') + \varepsilon(\tilde{u}_h')) = 0.723031$, $\overline{M} - M(\Omega) = 0.000253$. Lower bound $D(u_h') = 8.69506$, $\varepsilon(u_h') = 0.00693$; $\underline{M} = 2\pi/(D(u_h') + \varepsilon(u_h')) = 0.722039$, $\underline{M} - M(\Omega) = -0.000740$.
	Normal subdivision $(h = 0.025 \times \sqrt{2}$, $N(s) = 47136$, $N(q) = 20201)$ Upper bound $D(\tilde{u}_h') = 0.1150321$, $\varepsilon(\tilde{u}_h') = 0.0000118$; $\overline{M} = 2\pi(D(\tilde{u}_h') + \varepsilon(\tilde{u}_h')) = 0.722842$, $\overline{M} - M(\Omega) = 0.000063$. Lower bound $D(u_h') = 8.69359$, $\varepsilon(u_h') = 0.00173$; $\underline{M} = 2\pi/(D(u_h') + \varepsilon(u_h')) = 0.722594$, $\underline{M} - M(\Omega) = -0.000185$.

$N(s)$: Number of 2-simplices, $N(q)$: Number of 0-simplices.

Table 6 Modulus of a square domain with a long slit ($a = 0.999$).

Exact values	$D(u) = 20.4310$, $D(\tilde{u}) = 0.0489452$, $M(\Omega) = 2\pi/D(u) = 0.307532$.
Finite element approximations	Original triangulation ($h = 0.05 \times \sqrt{2}$, $N(s) = 16816$, $N(q) = 8619$) Upper bound $D(\tilde{u}_h') = 0.0489440$, $\varepsilon(\tilde{u}_h') = 0.0000109$; $\overline{M} = 2\pi(D(\tilde{u}_h') + \varepsilon(\tilde{u}_h')) = 0.307593$, $\overline{M} - M(\Omega) = 0.000061$. Lower bound $D(u_h') = 20.4333$, $\varepsilon(u_h') = 0.0082$; $\underline{M} = 2\pi/(D(u_h') + \varepsilon(u_h')) = 0.307374$, $\underline{M} - M(\Omega) = -0.000157$.
	Normal subdivision ($h = 0.025 \times \sqrt{2}$, $N(s) = 67264$, $N(q) = 34053$) Upper bound $D(\tilde{u}_h') = 0.0489449$, $\varepsilon(\tilde{u}_h') = 0.0000027$; $\overline{M} = 2\pi(D(\tilde{u}_h') + \varepsilon(\tilde{u}_h')) = 0.307547$, $\overline{M} - M(\Omega) = 0.000015$. Lower bound $D(u_h') = 20.4316$, $\varepsilon(u_h') = 0.0020$; $\underline{M} = 2\pi/(D(u_h') + \varepsilon(u_h')) = 0.307493$, $\underline{M} - M(\Omega) = -0.000039$.

$N(s)$: Number of 2-simplices, $N(q)$: Number of 0-simplices.

Table 7 Modulus of a square domain with a short slit ($a = 0.1$).

Exact values	$D(u) = 2.045656,\ D(\tilde{u}) = 0.488841,$ $M(\Omega) = 2\pi/D(u) = 3.071477.$
Finite element approximations	Original triangulation $(h = 0.05\times\sqrt{2},\ N(s) = 6424,\ N(q) = 3261)$ Upper bound $D(\tilde{u}'_h) = 0.488915,\ \varepsilon(\tilde{u}'_h) = 0.000275;$ $\overline{M} = 2\pi(D(\tilde{u}'_h) + \varepsilon(\tilde{u}'_h)) = 3.073667,$ $\overline{M} - M(\Omega) = 0.002190.$ Lower bound $D(u'_h) = 2.045456,\ \varepsilon(u'_h) = 0.000451;$ $\underline{M} = 2\pi/(D(u'_h) + \varepsilon(u'_h)) = 3.071100,$ $\underline{M} - M(\Omega) = -0.000377.$
	Normal subdivision $(h = 0.025\times\sqrt{2},\ N(s) = 25696,\ N(q) = 12945)$ Upper bound $D(\tilde{u}'_h) = 0.488859,\ \varepsilon(\tilde{u}'_h) = 0.000069;$ $\overline{M} = 2\pi(D(\tilde{u}'_h) + \varepsilon(\tilde{u}'_h)) = 3.072024,$ $\overline{M} - M(\Omega) = 0.000547.$ Lower bound $D(u'_h) = 2.045606,\ \varepsilon(u'_h) = 0.000113;$ $\underline{M} = 2\pi/(D(u'_h) + \varepsilon(u'_h)) = 3.071382,$ $\underline{M} - M(\Omega) = -0.000096.$

$N(s)$: Number of 2-simplices, $N(q)$: Number of 0-simplices.

Table 8 Modulus of a square domain with a short slit ($a = 0.001$).

Exact values	$D(u) = 0.818479$, $D(\tilde{u}) = 1.221779$, $M(\Omega) = 2\pi/D(u) = 7.676664$.
Finite element approxi-mations	Original triangulation ($h = 0.05 \times \sqrt{2}$, $N(s) = 15544$, $N(q) = 7821$) Upper bound $D(\tilde{u}'_h) = 1.221853$, $\varepsilon(\tilde{u}'_h) = 0.000275$; $\overline{M} = 2\pi(D(\tilde{u}'_h) + \varepsilon(\tilde{u}'_h)) = 7.678854$, $\overline{M} - M(\Omega) = 0.002190$. Lower bound $D(u'_h) = 0.818447$, $\varepsilon(u'_h) = 0.000072$; $\underline{M} = 2\pi/(D(u'_h) + \varepsilon(u'_h)) = 7.676287$, $\underline{M} - M(\Omega) = -0.000377$.
	Normal subdivision ($h = 0.025 \times \sqrt{2}$, $N(s) = 62176$, $N(q) = 31185$) Upper bound $D(\tilde{u}'_h) = 1.221798$, $\varepsilon(\tilde{u}'_h) = 0.000069$; $\overline{M} = 2\pi(D(\tilde{u}'_h) + \varepsilon(\tilde{u}'_h)) = 7.677211$, $\overline{M} - M(\Omega) = 0.000547$. Lower bound $D(u'_h) = 0.818471$, $\varepsilon(u'_h) = 0.000018$; $\underline{M} = 2\pi/(D(u'_h) + \varepsilon(u'_h)) = 7.676568$, $\underline{M} - M(\Omega) = -0.000095$.

$N(s)$: Number of 2-simplices, $N(q)$: Number of 0-simplices.